SPLIT BLOCK SUBDIVISION DOMINATION IN GRAPHS

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Abstract: A dominating set \( D \subseteq V[SB(G)] \) is a split dominating set in \([SB(G)]\). If the induced subgraph \((V[SB(G)] - D)\) is disconnected in \([SB(G)]\). The split domination number of \([SB(G)]\) is denoted by \( \gamma_{ssb}(G) \), is the minimum cardinality of a split dominating set in \([SB(G)]\). In this paper, some results on \( \gamma_{ssb}(G) \) were obtained in terms of vertices, blocks, and other different parameters of \( G \) but not members of \([SB(G)]\).

Further, we develop its relationship with other different domination parameters of \( G \).

Key words: Block graph, Subdivision block graph, split domination number.

INTRODUCTION

All graphs considered here are simple, finite, nontrivial, undirected and connected. As usual \( p, q and n \) denote the number of vertices, edges and blocks of a graph \( G \) respectively. In this paper, for any undefined term or notation can be found in F. Harary [3] and G. Chartrand and Ping Zhang [2]. The study of domination in graphs was begin by O. Ore [5] and C. Berge [1].

As usual, The minimum degree and maximum degree of a graph \( G \) are denoted by \( \delta(G) \) and \( \Delta(G) \) respectively. A vertex cover of a graph \( G \) is a set of vertices that covers all the edges of \( G \). The vertex covering number \( \alpha(G) \) is a minimum cardinality of a vertex cover in \( G \). The vertex independence number \( \beta(G) \) is the maximum cardinality of an independent set of vertices. A edge cover of \( G \) is a set of edges that covers all the vertices. The edge covering number \( \alpha_1(G) \) of \( G \) is minimum cardinality of an edge cover. The edge independence number \( \beta_1(G) \) of a graph \( G \) is the minimum cardinality of an independent set of edges.

A set of vertices \( D \subseteq V(G) \) is a dominating set. If every vertex in \( V - D \) is adjacent to some vertex in \( D \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of a dominating set in \( G \).

A dominating set \( D \) of a graph \( G \) is a split dominating set if the induced subgraph \((V - D)\) is disconnected. The split domination number \( \gamma_{s}(G) \) of a graph \( G \) is the minimum cardinality of a split dominating set. This concept was introduced by Kulli [4]. A dominating set \( D \) of \( G \) is a cototal dominating set if the induced subgraph \((V - D)\) has no isolated vertices. The cototal domination number \( \gamma_{cot}(G) \) of \( G \) is the minimum cardinality of a cototal dominating set. See [4]

The following figure illustrate the formation of \([SB(G)]\) of a graph \( G \).

The domination of split subdivision block graph is denoted by \( \gamma_{ssb}(G) \). In this paper, some results on \( \gamma_{ssb}(G) \) where obtained in terms of vertices, blocks and other parameters of \( G \).

We need the following Theorems for our further results:

MAIN RESULTS

Theorem A [4]: A split dominating set \( D \) of \( G \) is minimal for each vertex \( v \in D \), one of the following condition holds.

i) There exists a vertex \( u \in V - D \), such that \( N(u) \cap D = \{v\} \).

ii) \( v \) is an isolated vertex in \( D \).

iii) \( (V - D) \cup \{v\} \) is connected.

Theorem B [4]: For any graph , \( \gamma_{s}(G) \leq \frac{p \Delta(G)}{1 + \Delta(G)} \).

Now we consider the upper bound on \( \gamma_{ssb}(G) \) in terms of blocks in \( G \).

Theorem 2.1: For any graph \( G \) with \( n - \text{blocks and } n \geq 2 \), then \( \gamma_{ssb}(G) \leq n - 1 \).
Proof: For any graph $G$ with $n = 1$ block, a split domination does not exists. Hence we required $n \geq 2$ blocks. Let $S = \{B_1, B_2, B_3, \ldots, B_n\}$ be the number of blocks of $G$ and $M = \{b_1, b_2, b_3, \ldots, b_n\}$ be the vertices in $B(G)$ with corresponding to the blocks of $S$. Also $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices in $[SB(G)]$. Let $V_1 = \{v_1, v_2, v_3, \ldots, v_i\}$, $1 \leq i \leq n$, $V_1 \subset V$ be a set of cut vertices. Again consider a subset $V_1^* \subset V$ such that $V_1^* \notin N(V) \cap N(V_1)$ and $V_1 = V - V_1^*$. Let $V_2 = \{v_1, v_2, v_3, \ldots, v_s\}$, $1 \leq s \leq n$, $\forall v_s \in V$ which are not cut vertices such that $N(V_1) \cap N(V_2) = 0$. Then $V_1 \cup V_2$ is a dominating set.

Clearly $V[SB(G)] - \{V_1 \cup V_2\} = H$ is disconnected graph.

Then $V_1 \cup V_2$ is a $\gamma_{ssB}(G)$ which gives $\gamma_{ssB}(G) \leq n - 1$.

The following Theorem, we obtain an upper bound for $\gamma_{ssB}(G)$ in terms of vertices added to $B(G)$.

**Theorem 2.2:** For any connected $(p, q)$ graph with $n \geq 2$ blocks, then $\gamma_{ssB}(G) \leq R$ where $R$ is the number of vertices added to $B(G)$.

Proof: For any nontrivial connected graph $G$. If the graph $G$ has $n = 1$ block, then by the definition, split domination set does not exists. Hence $n \geq 2$ blocks. Let $S = \{B_1, B_2, B_3, \ldots, B_k\}$ be the blocks of $G$ and $M = \{b_1, b_2, b_3, \ldots, b_n\}$ be the vertices in $B(G)$ which corresponds to the blocks of $S$. Now we consider the following cases.

Case 1: Suppose each block of $B(G)$ is an edge. Then $R = q - 1 = E[B(G)]$. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices. Now consider $V_1 = \{v_1, v_2, v_3, \ldots, v_i\}$, $1 \leq i \leq n$ is a set of cut vertices.

Let $V_2 \subseteq V_1$, $\forall v_j \in V_2$ are adjacent to end vertices of $[SB(G)]$. Again there exists a subset $V_3$ of $V_1$ with the property $V[SB(G)] - \{V_2 \cup V_3\} = H$ where $\forall v_n \in H$ is adjacent to at least one vertex of $(V_2 \cup V_3)$ and $H$ is a disconnected graph. Hence $V_2 \cup V_3$ is a $\gamma_{ssB}(G)$ set of $G$. By Theorem 1,

$|V_2 \cup V_3| \leq R$.

Subcase 2.1: Assume $B(G) = \{K_1, K_2, K_3, \ldots, K_m\}$, then $V[SB(G)] - \{V[B(G)]\}$ where $\forall v_i \in q[B(G)]$ is an isolates. Hence $|q[B(G)]| \geq |V[B(G)]|$ which gives $\gamma_{ssB}(G) \leq R$.

Sub case 2.2: Assume every block of $B(G)$ is $K_p, p \geq 3$.

Let $B(G) = \{K_1, K_2, K_3, \ldots, K_m\}$ then $V[SB(G)] - \{V[B(G)]\}$ where $\forall v_i \in q[B(G)]$ is an isolates. Hence $|q[B(G)]| \geq |V[B(G)]|$ which gives $\gamma_{ssB}(G) \leq R$.

We establish an upper bound involving the Maximum degree $\Delta(G)$ and the vertices of $G$ for split block sub division domination in graphs.

**Theorem 2.3:** For any graph $G$ with $n \geq 2$ blocks, then $\gamma_{ssB}(G) \leq \frac{|p\Delta(G)|}{1+\Delta(G)}$.

Proof: For split domination. We consider the graphs with the property $n \geq 2$ blocks. Let $S = \{B_1, B_2, B_3, \ldots, B_n\}$ be the blocks of $G$ and $M = \{b_1, b_2, b_3, \ldots, b_n\}$ be the vertices in $B(G)$ corresponding to the blocks of $S$. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices in $[SB(G)]$. Let $D$ be a $\gamma_s$ set of $[SB(G)]$. By Theorem A, each vertex $v \in V$, there exist a vertex $u \in V[SB(G)] - D$ is a split dominating set in $[SB(G)]$. Thus $\gamma(G) \leq |V[SB(G)] - D| = |\gamma_s(G)| \leq P - \gamma_{ssB}(G)$. Since by Theorem B, $\gamma_s(G) \leq \frac{P\Delta(G)}{1+\Delta(G)}$ which gives $\gamma_{ssB}(G) \leq \frac{P\Delta(G)}{1+\Delta(G)}$.

The following lower bound relationship is between split domination in $[SB(G)]$ and vertex covering number in $B(G)$. 

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Theorem 2.4: For any graph $G$ with $n \geq 2$ blocks, then $\gamma_{ssb}(G) \geq \alpha_0[B(G)]$, where $\alpha_0$ is a vertex covering number of $B(G)$.

Proof: We consider only those graphs which are not $n = 1$. Let $S = \{B_1, B_2, B_3, \ldots, B_n\}$ be the blocks of $G$ which corresponds to the set $M = \{b_1, b_2, b_3, \ldots, b_n\}$ be the vertices in $B(G)$. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices in $[SB(G)]$ such that $M \subseteq V$. Again $D = \{v_1, v_2, v_3, \ldots, v_i\}$, $1 \leq i \leq n$, $D \subseteq V$ such that $N(v_i) \cap N(v_j) = \emptyset$, $\forall v_i, v_j \in D$. Hence $\langle V[SB(G)] - D \rangle$ is disconnected, which gives $|V[SB(G)] - D| = \gamma_{ssb}(G)$. Now $M_1 = \{b_1, b_2, b_3, \ldots, b_i\}, 1 \leq i \leq n$ and $M_1 \subseteq M$ and each edge in $B(G)$ is adjacent to at least one vertex in $M_1$. Clearly $|M_1| = \alpha_0[B(G)]$. Hence $|V[SB(G)] - D| \geq |M_1|$ which gives $\gamma_{ssb}(G) \geq \alpha_0[B(G)]$.

The following result gives a upper bound for $\gamma_{ssb}(G)$ in terms of domination and end blocks in $G$.

Theorem 2.5: For any connected graph $G$ with $n \geq 2$ blocks and $N - end blocks$, then $\gamma_{ssb}(G) \leq \gamma(G) + N$.

Proof: Suppose graph $G$ is a block. Then by definition, the split domination does not exists. Now assume $G$ is a graph with at least two blocks. Let $S = \{B_1, B_2, B_3, \ldots, B_n\}$ be the set of blocks in $G$ and $M = \{b_1, b_2, b_3, \ldots, b_n\}$ be the vertices in $B(G)$ which corresponds to the blocks of $G$. Now $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices in $[SB(G)]$. Suppose $D$ is a $\gamma_s - set$ in $[SB(G)]$ of $G$, whose vertex set is $V = \{v_1, v_2, v_3, \ldots, v_i\}$. Note that at least one $v_i \in S$. More over, any component of $V - S$ is of size at least two. Thus $D$ is minimal which gives $|D| = \gamma_{ssb}(G)$. Again $S_1 = \{u_1, u_2, u_3, \ldots, u_n\}$ be the vertices in $G$ and $D_1 = \{u_1, u_2, u_3, \ldots, u_i\}, 1 \leq i \leq n, D_1 \subset S_1$. Every vertex of $S_1 - D_1$ is adjacent to at least one vertex of $D_1$. Suppose there exists a vertex $v \in D_1$ such that every vertex of $D_1 - V_1$ is not adjacent to at least one vertex $u \in S_1 - \{D_1 - v\}$. Thus $|S_1 - D_1| = \gamma(G)$. Hence $|D| \leq |S_1 - D_1| + N$ which gives $\gamma_{ssb}(G) \leq \gamma(G) + N$.

Theorem 2.6: For any connected graph $G$ with $n \geq 2$ blocks then $\gamma_{ssb}(G) \geq \beta_0(G) - 1$, where $\beta_0(G)$ is the independence number of $G$.

Proof: By the definition of split domination, $n \neq 1$. Let $S = \{B_1, B_2, B_3, \ldots, B_n\}$ be the blocks of $G$ which corresponds to the vertices of the set $M = \{b_1, b_2, b_3, \ldots, b_n\}$ in $B(G)$. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices in $[SB(G)]$ such that $M \subseteq V$. Let $H = \{v_1, v_2, v_3, \ldots, v_s\}$ be the set of vertices in $G$. We have the following cases.

Case 1: Suppose $B(G)$ is a tree. Let $V_1^1 = \{v_1, v_2, v_3, \ldots, v_s\}$ be the cut vertices in $[SB(G)]$. Again $V_1^{11} = \{v_1, v_2, v_3, \ldots, v_t\}, 1 \leq t \leq s$ and $V_1^{11} \subset V_1^1$, were $\forall v_t \in V_1^{11}$. Then we consider $V_1^1, V_1^2$ and $V_1^4$, where $V_1^{11} = \{v_1, v_2, v_3, \ldots, v_t\} = V_1^2 \cup V_3 \cup V_4$ with the property that $N(v_i) \cap N(v_j) = \emptyset, \forall v_i \in V_1^2$ and $\forall v_j \in V_3$ and $V_1^4$ is a set of all end vertices in $[SB(G)]$. Again $\langle V[SB(G)] \rangle = J$ where every $v \in J$ is an isolates. Thus $|V_1^{11}| = \gamma_{ssb}(G)$.

Case 2: Suppose $B(G)$ is not a tree. Again we consider sub cases of case 2

Subcase 2.1: Assume $B(G)$ is a block. Then in $[SB(G)], V[SB(G)] = V[B(G)] + \{k\}$, where $\forall k, degk = 2$. Thus $|K| = P_0$ the number of isolates in $V[SB(G)] - V[B(G)]$. Hence $|V[B(G)]| = \gamma_{ssb}(G)$. One can see that for the $\beta_0 - set$ as in case 1. We have $|V[B(G)]| \geq \beta_0 - 1$ which gives $\gamma_{ssb}(G) \geq \beta_0(G) - 1$.

Sub case 2.2: Assume $B(G)$ has at least two blocks. Then as in subcase 2.1, we have $\gamma_{ssb}(G) \geq \beta_0(G) - 1$.

The next result gives a lower bound on $\gamma_{ssb}(G)$ in terms of the diameter of $G$.

Theorem 2.7: For any graph $G$ with $n \geq 2$ blocks, then $\gamma_{ssb}(G) \geq \text{diameter}(G) - 2$. 
Proof : Suppose \( S = \{B_1, B_2, B_3, \ldots, B_n\} \) be the blocks of \( G \). Then \( M = \{b_1, b_2, b_3, \ldots, b_n\} \) be the corresponding block vertices in \( B(G) \). Suppose \( A = \{e_1, e_2, e_3, \ldots, e_k\} \) be the set of edges which constitutes the diametrical path in \( G \). Let \( S_1 = \{B_i\} \) where \( 1 \leq i \leq n, S_1 \subset S \). Suppose \( \forall B_i \in S_1 \) are non end blocks in \( G \), which gives cut vertices in \( B(G) \) and \( [SB(G)] \). Suppose \( V = \{v_1, v_2, v_3, \ldots, v_n\} \) be the vertices in \( [SB(G)] \). Again \( V_1 = \{v_1, v_2, v_3, \ldots, v_i\} \) where \( 1 \leq i \leq n \) such that \( V_1 \subset V \) then \( \forall v_i \in V_1 \) are cut vertices in \( [SB(G)] \). Since they are non end blocks in \( [SB(G)] \). Then \( V_1 \) is a \( \gamma_2 \) - set of \( [SB(G)] \). Clearly \( |V_1| = \gamma_{ssb}(G) \).

Suppose \( G \) is cyclic then there exists atleast one block \( B \) which contains a block diametrical path of length atleast two. In \( B(G) \) the block \( B \in V[B(G)] \) as a singleton and if almost two elements of \( \{A\} \in diameter \) of \( G \) then \( |A| - 2 \leq |V_1| \) gives \( \gamma_{ssb}(G) \geq \text{diameter}(G) - 2 \). Suppose \( G \) is acyclic then each edge of \( G \) is a block of \( G \). Now \( \forall B_i \in S, \exists e_i, e_j \in \{A\} \) where \( 1 \leq \{i, j\} \leq k \) gives \( \text{diameter}(G) - 2 \leq |V_1| \). Clearly we have \( \gamma_{ssb}(G) \geq \text{diameter}(G) - 2 \).

The following result is a relationship between \( \gamma_{ssb}(G) \), domination and vertices of \( G \).

**Theorem 2.8**: For any graph \( G \) with \( n \geq 2 \) blocks then \( \gamma_{ssb}(G) + \gamma(G) \leq P + 1 \).

Proof: Suppose the graph \( G \) has one block, then split domination does not exists. Hence \( n \geq 2 \) blocks.

Suppose \( S = \{B_1, B_2, B_3, \ldots, B_n\} \) be the blocks of \( G \). Then \( M = \{b_1, b_2, b_3, \ldots, b_n\} \) be the corresponding block vertices in \( B(G) \). Let \( H = \{v_1, v_2, v_3, \ldots, v_n\} \) be the set of vertices in \( G \). Also \( J = \{v_1, v_2, v_3, \ldots, v_i\} \) where \( 1 \leq i \leq n \) such that \( J \subset H \) and \( \forall v_i \in H - J \) is adjacent to atleast one vertex of \( J \). Hence \( |J| = \gamma(G) \). Let \( V = \{v_1, v_2, v_3, \ldots, v_s\} \) be the set of vertices in \( [SB(G)] \). Now \( S_1 = \{B_i\} \) where \( 1 \leq i \leq n, S_1 \subset S \) and \( \forall B_i \in S_1 \) are non end blocks in \( G \). Then we have \( V_1 \subset V \) which corresponds to the elements of \( S[S_1] \) such that \( V_1 \) forms a minimal dominating set of \( [SB(G)] \). Since each element of \( V_1 \) is a cut vertex, then

Next, the following upper bound for split domination in \( [SB(G)] \) is in terms of edge covering number of \( G \).

**Theorem 2.9**: For any connected \((p, q)\) graph with \( n \geq 2 \) blocks, then \( \gamma_{ssb}(G) \leq \alpha_1(G) + 1 \) where \( \alpha_1(G) \) is the edge covering number.

Proof: For any non trivial connected graph \( G \) with \( n = 1 \) block, then by definition of split domination, the split domination set does not exists. Hence \( n \geq 2 \) blocks.

Let \( S = \{B_1, B_2, B_3, \ldots, B_n\} \) be the blocks of \( G \) which corresponds to the set \( M = \{b_1, b_2, b_3, \ldots, b_n\} \) be the vertices in \( B(G) \). Let \( V = \{v_1, v_2, v_3, \ldots, v_n\} \) be the vertices in \( [SB(G)] \) such that \( M \subset V \). We have the following cases.

Case 1: Suppose each block is an edge in \( G \). Then \( E(G) = \{E_1(G) \cup E_2(G)\} \) where \( E_1(G) \) is the set of end edges. If every cut vertex of \( G \) is adjacent with an end vertex. Then \( E_1(G) \) and \( E_2(G) \). If \( E_2(G) = \emptyset \). Then \( |E_1(G)| = \alpha_1(G) \). Otherwise \( |E_1(G) \cup E_2(G)| = \alpha_1(G) \).

Let \( D_1 = \{v_s\}, 1 \leq s \leq n \) and \( D_1 \subset V \), then there exist atleast one cut vertices in \( [SB(G)] \). Let \( D_2 = \{v_t\}, 1 \leq t \leq n \) and \( D_2 \subset V \) which are non cut vertices in \( [SB(G)] \). Again \( D_1^1 = \{v_s\}, 1 \leq l \leq t \) and \( D_1^1 \subset D_2 \). The \( N(D_1^1) \cap N(v_t) = \emptyset \) then \( (D_1^1 \cup D_2) \) is a split dominating set. Hence \( |V [SB(G)] - (D_1^1 \cup D_2)| = \gamma_{ssb}(G) \). Since \( |V [SB(G)] - (D_1^1 \cup D_2)| \) has more than one component. Hence \( |V [SB(G)] - (D_1^1 \cup D_2) | \leq \alpha_1(G) + 1 \) which gives \( \gamma_{ssb}(G) \leq \alpha_1(G) + 1 \).

Case 2: Suppose \( G \) has atleast one block which is not an edge. Let \( D_1 = \{v_1, v_2, v_3, \ldots, v_i\}, 1 \leq i \leq n \) and \( D_1 \subset V \) be the set of cut vertices such that \( N(v_i) \neq \emptyset \). Again \( D_2 = \{v_1, v_2, v_3, \ldots, v_i\}, 1 \leq i \leq t \) be the set of cut vertices in \( [SB(G)] \) such that \( N(v_i) \cap N(v_t) = \emptyset \). \( N(v_t) \cap N(v_i) = v_k, \) where \( v_i, v_t, v_k \in D \) and \( v_k \in V [SB(G)] - D \). Hence \( |V [SB(G)] - D| = \gamma_{ssb}(G) \). As in case 1, \( \alpha_1(G) \) will increase. Hence \( |V [SB(G)] - D| \leq \alpha_1(G) + 1 \) which gives \( \alpha_1(G) + 1 \geq \gamma_{ssb}(G) \).
The following lower bound for split domination in \([SB(G)]\) is in terms of edge independence number in \([B(G)]\).

**Theorem 2.10**: For any graph \(G\) with \(n \geq 2\) blocks then \(\gamma_{ssb}(G) \geq \beta_1[B(G)]\).

Proof: By the definition of Split domination, we need \(n \geq 2\) blocks. We have the following cases.

Case 1: Suppose each block in \([B(G)]\) is an edge. Let \(E = \{e_1, e_2, e_3, \ldots, e_n\}\) be the set of edges in \([B(G)]\). Also \(E_1 = \{e_s\}, 1 \leq s \leq n\) be a set of alternative edges in \([B(G)]\). Then \(|E_1| = \beta_1[B(G)]\).

Consider \(V = \{v_1, v_2, v_3, \ldots, v_n\}\) be the vertices in \([SB(G)]\), again \(V_1 = \{v_1, v_2, v_3, \ldots, v_i\}\) be the cut vertices which are adjacent to at least one vertex of \(E_1\) and \(V_2 = \{v_s\}\) are the end vertices in \([SB(G)]\). Further \((V[SB(G)] - (V_1 \cup V_2))\) is disconnected. Then \(|V_1 \cup V_2|\) is a \(\gamma_{ssb}\) set.

Hence \(|V_1 \cup V_2| \geq |E_1|\) which gives \(\gamma_{ssb}(G) \geq \beta_1[B(G)]\).

Case 2: Suppose there exists at least one block which is not an edge. Let \(E = \{e_1, e_2, e_3, \ldots, e_n\}\) be the set of edges in \([B(G)]\). Again \(E_1 = \{e_s\}, 1 \leq s \leq n\) is the set of alternative edges in \([B(G)]\) which gives \(|E_1| = \beta_1[B(G)]\).

Suppose \(V = \{v_1, v_2, v_3, \ldots, v_n\}\) be the vertices of \([SB(G)]\). Then \(V = V_1 \cup V_2\) where \(V_1\) is a set of cut vertices and \(V_2\) is a set of non cut vertices. Now we consider \(V_1 = V_1^1 \subset V_1\) and \(V_2 = V_2^1 \subset V_2\) such that \((V[SB(G)] - (V_1^1 \cup V_2^1))\) has more than one component. Hence \(\{V_1^1 \cup V_2^1\}\) is a \(\gamma_{ssb}\) set and \(|V_1^1 \cup V_2^1| \geq \beta_1[B(G)]\) which gives \(\gamma_{ssb}(G) \geq \beta_1[B(G)]\).

In the following theorem, we expressed the lower bound for \(\gamma_{ssb}(G)\) in terms of cut vertices of \([B(G)]\).

**Theorem 2.11**: For any connected graph \(G\) with \(n \geq 2\) blocks then \(\gamma_{ssb}(G) \geq C[B(G)]\) where \(C\) is the cut vertices in \([B(G)]\).

Proof: Suppose graph \(G\) is a block. Then by the definition, of split domination, \(n \geq 2\). consider the following cases.

Case 1: Suppose each block of \([B(G)]\) is an edge. Then we consider \(S = \{v_1, v_2, v_3, \ldots, v_n\}\) be the cut vertices in \([B(G)]\). Now \(V = \{v_1, v_2, v_3, \ldots, v_n\}\) be the vertices in \([SB(G)]\).

The following result gives an lower bound on \(\gamma_{ssb}(G)\) in terms of \(\gamma_{cot}(G)\).

**Theorem 2.12**: For any nontrivial tree with \(n \geq 2\) blocks, \(\gamma_{ssb}(G) \geq \gamma_{cot}(G) - 1\).

Proof: We consider only those graphs which are not \(n = 1\). Let \(H = \{v_1, v_2, v_3, \ldots, v_p\}\), \(H_1 = \{v_1, v_2, v_3, \ldots, v_i\}, 1 \leq i \leq p\) be a subset of \(V(G) = H\) which are end vertices in \(G\). Let \(J = \{v_1, v_2, v_3, \ldots, v_j\} \subseteq V(G)\) with \(1 \leq j \leq p\) such that \(\forall v_j \in J\), \(N(v_i) \cap N(v_j) = \emptyset\) and \((V(G) - (H_1 \cup J))\) has no isolates, then \(|H_1 \cup J| = \gamma_{cot}(G)\). Let \(V = \{v_1, v_2, v_3, \ldots, v_n\}\) be the vertices in \([SB(G)]\). Consider \(D = \{v_1, v_2, v_3, \ldots, v_i\} = V_1 \cup V_2 \cup V_3\) be the set of all vertices of \([SB(G)]\). Where \(\forall v_k \in V_1\) and \(v_k \in V_2\) with the property \((v_k) \cap N(v_k) = \emptyset\). \(V_1 \subseteq V_3\) is a set of all end vertices in \([SB(G)]\). The \(D\) is an isolates. \(|D|\) gives minimum split domination in \([SB(G)]\).

Thus \(|D| = \gamma_{ssb}(G)\). Clearly \(|H_1 \cup J| - 1 \leq |D|\) which gives \(\gamma_{ssb}(G) \geq \gamma_{cot}(G) - 1\).

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