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COUPLED FIXED POINT THEOREMS FOR α - ψ CONTRACTIVE TYPE MAPPINGS IN PARTIALLY ORDERED PARTIAL B - METRIC SPACES

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***Abstract*— Sastry et.al.[28] used the concept of $\alpha-\psi$ contraction on a complete partially ordered partial b - metric space to study the existence and uniqueness of coupled fixed points for nonlinear contractive mappings controlled by a generalized contractive type condition. In this connection, an open problem is posed. In this paper, we partially answer the open problem in the affirmative. Still the open problem awaits complete answer. Supporting example is also provided.**

***Index Terms*—Component, formatting, style, styling, insert. Coupled fixed point, $\alpha-\psi$ contractive mappings, mixed monotone property, α admissible, partial b - metric, complete partially ordered partial b - metric space.**

I. INTRODUCTION

Most of the fixed point theorems in nonlinear analysis usually start with Banach [7] contraction principle. A huge amount of literature is witnessed on applications, generalizations and extensions of this principle carried out by several authors in different directions like weakening the hypothesis and considering different mappings. Fixed point theory is an essential tool in the study of various varieties of problems in control theory, economic theory, nonlinear analysis and global analysis. But all the generalizations may not be from this principle. In 1989, Bakktin [6] introduced the concept of a b - metric space as a generalization of a metric space. In 1993, Czerwik[9] extended many results related to the b - metric space. In 1994, Matthews [17] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, O'Neill [26] generalized the concept of partial metric space by admitting negative distances. In 2013, Shukla [32] generalized both the concepts of b - metric and partial metric space by introducing the notation of partial b - metric spaces. Many authors recently studied the existence of fixed points of self maps in different types of metric spaces [14,33,20,30,35]. Some authors [4,18,22,28,29] obtained some fixed point theorems in b - metric spaces. After that some authors started to prove $\alpha-\psi$ ~ versions of certain fixed point theorems in different types of metric spaces [3,14]. Recently Samet et.al [27] and Jalal Hassanzadeasl [12] obtained fixed point theorems for $\alpha-\psi$ contractive mappings. Mustafa [22] gave a generalization of Banach contraction principle in complete ordered partial b - metric space by introducing the notion of a generalized $\alpha-\psi$ weakly contractive mapping. Bhaskar and Lakshmikantham [5] introduced the notation of mixed monotone property and proved coupled fixed point theorems. In 2012, Mohammad Mursaleen et.al [19] proved coupled fixed point theorems for α - ψ contractive type mappings in partially ordered metric spaces. Sastry et.al.[28] discussed about $\alpha-\psi$ contraction on a complete partially ordered partial metric space to study the existence and uniqueness of coupled fixed points and posed an open problem to discuss the same on a complete partially ordered partial b- metric space.

II. TYPE STYLE AND FONTS

In this paper we take up this open problem of Sastry et.al.[28] to study the existence and uniqueness of coupled fixed points for contractive mappings controlled by a generalized contractive type condition on a complete partially ordered partial b- metric space under α -ψ contraction for $s \geq 1$ and gave a partial answer for this problem. In fact, we show that coupled fixed point theorems on a complete partially ordered partial b - metric space with coefficient of partial b - metric space $s \geq 1$ by using mixed monotone property under α -ψ contraction. A supporting example is also given. Further an open problem is also given at the end of this paper.

Definition 1.1.(H.Aydi et al [4]) Let X be a non empty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is called a b - metric if for all $x, y, z \in X$ the following conditions are satisfied.

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq s\{d(x, z) + d(z, y)\}$

Definition 1.2. (S.G.Matthews [17]) Let X be a non empty set. A function $p : X \times X \to [0,\infty)$ is called a partial metric, if for all $x, y, z \in X$ the following conditions are satisfied.

(i) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$.

(ii) $p(x, x) \leq p(x, y)$

(iii) $p(x, y) = p(y, x)$

(iv) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$

The pair (X, p) is called a partial metric space.

Shukla [32] introduced the notation of a partial b - metric space as follows.

Definition 1.3.(S.Shukla[32]) Let X be a non empty set and let $s \geq 1$ be a given real number. A function $p: X \times X \rightarrow [0,\infty)$ is called a partial b - metric if for all $x, y, z \in X$ the following conditions are satisfied.

(i) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$

(ii) $p(x,x) \leq p(x,y)$

(iii) $p(x,y) = p(y,x)$

(iv) $p(x,y) \leq s\{p(x,z)+p(z,y)\} - p(z,z)$

The pair (X, p) is called a partial b - metric space. The number $s \geq 1$ is called a coefficient of (X, p). We note that if $s = 1$, we get definition 1.2

Definition 1.4. A sequence $\{x_n\}$ in a partial b - metric space (X, p) is said to be:

(i) convergent to a point $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x,x)$

(ii) a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite

(iii) a partial b - metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$

such that $\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x,x)$.

Definition 1.5.(E.Karapinar and B.Samet [15]) Let $(X, \leq)$ be a partially ordered set. A sequence $\{x_n\} \in X$ is said to be non decreasing, if $x_n \leq x_{n+1} \forall n \in N$

In Sastry et.al[28], the notion of a partially ordered partial metric space is introduced.

Definition 1.6.(Z.Mustafa[20]) A triple $(X, \leq, p)$ is called an ordered partial b - metric space if $(X, \leq)$ is a partially ordered set and p is a partial b - metric on X. For definiteness sake Sastry et.al[28](Definition 2.1) adopting the definition 1.2 for partial metric and defined the triple $(X, \leq, p)$ as partially ordered partial metric space. A partially ordered partial metric space $(X, \leq, p)$ is said to be complete if every Cauchy sequence is convergent.

Definition 1.7.(Mohammad Mursaleen. et. al. [19]) Define $\Psi = \{\psi / \psi : [0,\infty) \rightarrow [0,\infty)$ is non-decreasing and satisfies $(1.7.1)\}$

$$\lim_{r\to t^+} \psi(r) < t \quad \forall t > 0 \qquad 1.7.1)$$

Definition 1.8.(T.G.Bhaskar and V.Lakshmikantham [5]) Let $(X, \leq)$ be a partially ordered set and $F: X \times X \rightarrow X$ be a mapping. Then F is said to have the mixed monotone property if $F(x, y)$ is monotone non - decreasing in x and is monotone non - increasing in y; i.e., for any $x, y \in X$,

$$x_1, x_2 \in X \;,\; x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y) \qquad (1.8.1)$$

and $y_1, y_2 \in X \;\; y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2) \qquad (1.8.2)$

Definition 1.9.(T.G.Bhaskar and V.Lakshmikantham [5]) An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F: X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$

Definition 2.0.(Aiman Mukheimer[18]) Let (X, p) be a partial b - metric space, $\alpha: X \times X \rightarrow [0,\infty)$ and $T: X \rightarrow X$ be a given mappings. We say that T is α - admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$. Also we say that T is L_α - admissible, or R_α - admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, y) \geq 1$ or $\alpha(x, Ty) \geq 1$ respectively.

Definition 2.1.(Mohammad Murasaleen et.al. [19]) Let $F: X \times X \rightarrow X$ and $\alpha: X^2 \times X^2 \rightarrow [0,\infty)$ be two mappings. Then F is said to be

α - admissible if, $\alpha\{(x, y), (u, v)\} \geq 1 \Rightarrow$
$\alpha(\{F(x, y), F(y, x)\}, \{F(u, v), F(v, u)\}) \geq 1$

for all $x, y, u, v \in X$

Using the definition 1.6, the following theorems are established in Sastry et.al.[28].

Theorem 2.2.(Sastry et.al.[28] Theorem 2.7)Let $(X, \leq, p)$ be a complete partially ordered partial metric space. Let $F: X \times X \rightarrow X$ be a mapping having the mixed monotone property of X .Suppose $\psi \in \Psi$ and $\alpha: X^2 \times X^2 \rightarrow [0,\infty)$ such that for $x, y, u, v \in X$, the following holds :

$$\alpha((x, y), (u, v)) \; p(F(x, y), F(u, v)) \leq \psi(max\{p(x,u), p(y,v)\}) \;\; \forall x \geq u \, and \; y \leq v \qquad (2.2.1)$$

Suppose that

(i) F is α - admissible.

(ii)there exists $x_0, y_0 \in X$ such that

$\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1,$
$\alpha((y_0, x_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1$

and

$x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$.

(iii) F is continuous Then F has a coupled fixed point, that is $\exists x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$

Theorem 2.3.(Sastry et.al.[28] Theorem 2.8) Let $(X, \leq, p)$ be a complete partially ordered partial metric space. Let $F: X \times X \rightarrow X$ be a mapping having the mixed monotone property of X. Suppose $\psi \in \Psi$ and $\alpha: X^2 \times X^2 \rightarrow [0,\infty)$ such that for $x, y, u, v \in X$, the following holds :

$$\alpha((x, y), (u, v)) \; p(F(x, y), F(u, v)) \leq \psi(max\{p(x,u), p(y,v)\}) \;\; \forall x \geq u, y \leq v. \qquad (2.3.1)$$

Suppose that

(i) F is α - admissible.

(ii) there exist $x_0, y_0 \in X$ such that $\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))) \geq 1$,
$\alpha((y_0,x_0),(F(y_0,x_0),F(x_0,y_0))) \geq 1$.
and $x_0 \leq F(x_0,y_0)$ and $y_0 \leq F(y_0,x_0)$
(iii)If $\{x_n\}$ is a non-decreasing sequence and $\{y_n\}$ is a non-increasing sequence, then $\lim_{n\to\infty} x_n = x \Rightarrow x_n \leq x$ and $\lim_{n\to\infty} y_n = y \Rightarrow y_n \geq y$
(iv) Further If $\alpha((x_n,y_n),(x_{n+1},y_{n+1})) \geq 1$ and $\alpha((y_n,x_n),(y_{n+1},x_{n+1})) \geq 1 \quad \forall n$, then $\alpha((x_n,y_n),(x,y)) \geq 1$ and $\alpha((y_n,x_n),(y,x)) \geq 1$
Then F has a coupled fixed point, that is $\exists x, y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$.

Theorem 2.4. (Sastry et.al.[28] Theorem 2.9)Let $(X, \leq, p)$ be a complete partially ordered partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property of X. Suppose $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \to [0,\infty)$ such that for $x,y,u,v \in X$, the following holds :

$$\alpha((x,y),(u,v))\ p(F(x,y),F(u,v)) \leq \psi(max\{p(x,u), p(y,v)\}) \qquad (1.14.1)$$

$\forall x \geq u\, and\ y \leq v$. Suppose that
(i) F is α - admissible.
(ii) there exist $x_0, y_0 \in X$ such that $\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))) \geq 1$,
$\alpha((y_0,x_0),(F(y_0,x_0),F(x_0,y_0))) \geq 1$
and
$x_0 \leq F(x_0,y_0)$ and $y_0 \geq F(y_0,x_0)$.
(iii) F is continuous
Suppose for every (x,y) and (s,t) in $X \times X$, there exists (u,v) such that $\alpha((x,y),(u,v)) \geq 1$ and $\alpha((s,t),(u,v)) \geq 1$.
Then F has a unique coupled fixed point.

Theorem 2.5.(Sastry et.al.[28] Theorem 2.10)Let $(X, \leq, p)$ be a complete partially ordered partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property of X. Suppose $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \to [0,\infty)$ such that for $x,y,u,v \in X$, the following holds:

$$\alpha((x,y),(u,v))\ p(F(x,y),F(u,v)) \leq \psi(max\{p(x,u), p(y,v)\}) \qquad (2.5.1)$$

$\forall x \geq u\, and\ y \leq v$. Suppose that
(i) F is α - admissible.
(ii) there exist $x_0, y_0 \in X$ such that $\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))) \geq 1$, $\alpha((y_0,x_0),(F(y_0,x_0),F(x_0,y_0))) \geq 1$.and
$x_0 \geq F(x_0,y_0)$ and $y_0 \leq F(y_0,x_0)$.
(iii) F is continuous
(iv)If sequence $\{x_n\}$ is non-increasing and sequence $\{y_n\}$ is non-decreasing then $\lim_{n\to\infty} x_n = x \Rightarrow x_n \geq x$ and $\lim_{n\to\infty} y_n = y \Rightarrow y_n \leq y$;
Further If $\alpha((x_n,y_n),(x_{n+1},y_{n+1})) \geq 1$ and $\alpha((y_n,x_n),(y_{n+1},x_{n+1})) \geq 1\ \forall n$,then $\alpha((x_n,y_n),(x,y)) \geq 1$ and $\alpha((y_n,x_n),(y,x)) \geq 1$
Then F has a coupled fixed point, that is $\exists x, y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$.
In this connection, the following open problem is posed in Sastry et.al.[28]. Are theorems 2.2, 2.3, 2.4, 2.5 hold good in partially ordered partial b - metric spaces ($s \geq 1$)?
In the next section we partially answer the open problem by imposing a link between coefficient s>1of a partially ordered partial b - metric space $(X, \leq, p)$ and the function $\psi \in \Psi$.
As theorems 1.12, 1.13, 1.14, 1.15 proved when $s = 1$ in Sastry et.al [28], we assume, without laws of generality, that s > 1 .
Main Result In this section we continue our study of the open problem of Sastry et.al.[28] under $\alpha - \psi$ contractions. We study the existence and uniqueness of coupled fixed point theorems by considering maps on complete partially ordered partial b - metric space under α - ψ contraction. This paper is a sequel of Sastry et.al.[28]. We begin this section with the following definition.
Definition 2.6.(Sastry et.al.[29]) Suppose $(X, \leq)$ is a partially ordered set and is a partial b – metric p as in definition 1.3 with coefficient $s \geq 1$. Then we say that the triplet $(X, \leq, p)$ is a partially ordered partial b - metric space. A partially ordered partial b - metric space $(X, \leq, p)$ is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.7. Let $(X, \leq, p)$ be a partially ordered partial b metric space, with coefficient $s \geq 1$.
Define Ψ_s = $\{\psi / \psi : [0,\infty) \to [0,\infty)$ is non-decreasing and satisfies (2.7.1)$\}$

$$\lim_{r\to t^+} \psi(r) < \frac{t}{s} \quad \forall t > 0 \qquad (2.7.1)$$

We observe that Ψ_1 = Ψ. In what follows, we assume, unless otherwise specified, that $(X, \leq, p)$ is a partially ordered partial b - metric space with s > 1 .
Now we state the following useful lemmas, whose proofs can be found in Sastry. et. al [29].
Lemma 2.8. Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space.

Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$ Then $\lim_{n\to\infty} p(x_n, x)$ = $\lim_{n\to\infty} p(x_n, y)$ = $p(x, y)$ and hence $x = y$

Lemma 2.9. $p(x, y) = 0 \Rightarrow x = y$

Lemma3.0. If $\psi \in \Psi_s$ then (i) $\lim_{n\to\infty} \psi^n(t) = 0 \ \forall t > 0$

(ii) $\psi(t) < \frac{t}{s} \forall t > 0$ and $s \geq 1$ is the coefficient of (X, p)

Now we state our first main result:

Theorem 3.1. Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient \$ s > 1 \$. Let $F : X \times X \to X$ be a mapping having the mixed monotone property of X. Suppose $\psi \in \Psi_s$ and $\alpha : X^2 \times X^2 \to [0, \infty)$ such that for $x, y, u, v \in X$, the following holds :

$$\alpha((x, y), (u, v)) \quad p(F(x, y), F(u, v)) \leq \psi(max\{p(x, u), p(y, v)\}) \tag{3.1.1}$$

$\forall x \geq u \; and \; y \leq v$ .Suppose that

(i) F is α - admissible.

(ii) there exist $x_0, y_0 \in X$ such that $\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1$, $\alpha((y_0, x_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1$ and $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$.

(iii) F is continuous

Then F has a coupled fixed point, that is $\exists x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$

Proof: Let $x_0, y_0 \in X$ be as in (ii). Then $\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1$, $\alpha((y_0, x_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1$, $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$.

Write $F(x_0, y_0) = x_1$ and $F(y_0, x_0) = y_1$.

Then clearly $x_0 \leq x_1$ and $y_0 \geq y_1$.

Define $x_2, y_2 \in X$ as $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$. Now from (ii),

$\alpha((x_0, y_0), (x_1, y_1)) \geq 1$

$\Rightarrow \alpha((F(x_0, y_0), F(y_0, x_0)), (F(x_1, y_1), F(y_1, x_1))) \geq 1$

$\Rightarrow \ \alpha((x_1, y_1), (x_2, y_2)) \geq 1$

By mixed monotone property $x_0 \leq x_1$ and $y_0 \geq y_1$

$\Rightarrow F(x_0, y_0) \leq F(x_1, y_0) \leq F(x_1, y_1)$ and $F(y_0, x_0) \geq F(y_1, x_0) \geq F(y_1, x_1)$

$\Rightarrow$ $x_1 \leq x_2$ and $y_1 \geq y_2$. Inductively, define $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$, $n = 0, 1, 2, ...$

Suppose $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$. Then by mixed monotone property,

$x_{n+1} = F(x_n, y_n) \leq F(x_{n+1}, y_n) \leq F(x_{n+1}, y_{n+1}) = x_{n+2}$
and
$y_{n+1} = F(y_n, x_n) \geq F(y_{n+1}, x_n) \geq F(y_{n+1}, x_{n+1}) = y_{n+2}$

$\therefore \ x_{n+1} \leq x_{n+2}$ and $y_{n+1} \geq y_{n+2}$

Hence sequence $\{x_n\}$ is non-decreasing and sequence $\{y_n\}$ is non-increasing.

Suppose $(x_n, y_n) = (x_{n+1}, y_{n+1})$ for some n.

Then $x_n = x_{n+1} = F(x_n, y_n)$ and $y_n = y_{n+1} = F(y_n, x_n)$

$\therefore$ F has a coupled fixed point. In fact (x_n, y_n) is a coupled fixed point.

Hence we may assume that $(x_{n+1}, y_{n+1}) \neq (x_n, y_n) \ \forall n \geq 0$.

Since F is α - admissible, we have

$\alpha \quad ((x_0, y_0), (x_1, y_1)) \quad \geq 1 \quad \Rightarrow \quad \alpha$ $((F(x_0, y_0), F(y_0, x_0)), (F(x_1, y_1), F(y_1, x_1))) \ \geq 1$

$\Rightarrow \alpha \ ((x_1, y_1), (x_2, y_2)) \ \geq 1$

$\therefore$ By Mathematical Induction

$$\alpha \ ((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1 \ \ \forall n \geq 0 \tag{3.1.2}$$

Similarly $\alpha \ ((y_n, x_n), (y_{n+1}, x_{n+1})) \ \geq 1 \ \ \forall n \geq 0$ (3.1.3)

Now $p \ (x_n, x_{n+1}) = p \ (F(x_{n-1}, y_{n-1}), F(x_n, y_n))$

$\leq \alpha \ ((x_{n-1}, y_{n-1}), (x_n, y_n)) \ \ p \ (F(x_{n-1}, y_{n-1}), F(x_n, y_n))$

$\leq \ \psi(max \ \{p \ (x_{n-1}, x_n), p \ (y_{n-1}, y_n)\})$

Similarly $p \ (y_n, y_{n+1}) \leq$

$\psi(max \ \{p \ (y_{n-1}, y_n), p \ (x_{n-1}, x_n)\})$

$\therefore \quad max \ \{p \ (x_n, x_{n+1}), p \ (y_n, y_{n+1})\} \leq \psi(max \ \{p \ (x_{n-1}, x_n), p \ (y_{n-1}, y_n)\})$

$\Rightarrow \ \lambda_n \ \leq \ \psi(\lambda_{n-1}) < \frac{1}{s} \lambda_{n-1}$ (by (ii) of lemma 3.0)

where λ_n = $max \ \{p \ (x_n, x_{n+1}), p \ (y_n, y_{n+1})\}$

$\therefore \ \ \lambda_n \ \leq \ \psi(\lambda_{n-1}) \ \leq \ \psi^2(\lambda_{n-2}) \ \leq ... \ \leq \ \psi^n(\lambda_0)$

where λ_0 \$=\$ $p \ (x_0, x_1) > 0$

$\therefore$ sequence $\{\lambda_n\}$ is decreasing and $\lim_{n\to\infty} \lambda_n \leq \lim_{n\to\infty} \psi^n(\lambda_0) = 0$ (by (i) lemma 3.0)

$\therefore$ sequence $\{\lambda_n\}$ is convergent and converges to 0

$\therefore \lim_{n\to\infty} \lambda_n = 0$

$$\Rightarrow \lim_{n\to\infty} max \ \{p \ (x_n, x_{n+1}), p \ (y_n, y_{n+1})\} \ = 0 \tag{3.1.4}$$

$\Rightarrow \ \lim_{n\to\infty} \ p \ (x_n, x_{n+1}) = 0$

$$\text{and } \lim_{n\to\infty} \ p \ (y_n, y_{n+1}) \ = 0 \tag{3.1.5}$$

Let us show that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

If possible let $\{x_n\}$ or $\{y_n\}$ fails to be a Cauchy sequence, Then either

$\lim\limits_{m,n\to\infty} p(x_m, x_n) \neq 0$ or $\lim\limits_{m,n\to\infty} p(y_m, y_n) \neq 0$

$\Rightarrow max\{\lim\limits_{m,n\to\infty} p(x_m, x_n), \lim\limits_{m,n\to\infty} p(y_m, y_n)\} \neq 0$

$\therefore \exists \epsilon > 0$ for which we can find sub sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k \ni$ $max\{p(x_{m_k}, x_{n_k}), p(y_{m_k}, y_{n_k})\} \geq \epsilon$.

Further we can choose n_k to be the smallest positive integer $\ni$ $\{m_k\}$ and $\{n_k\}$ satisfy $max\{p(x_{m_k}, x_{n_k}), p(y_{m_k}, y_{n_k})\} \geq \epsilon$ and max $p(x_{m_k}, x_{n_k-1}), p(y_{m_k}, y_{n_k-1})\} < \epsilon$

Now we explore the properties of the sequences which use in the sequel.

I. $\lim\limits_{k\to\infty}(max\{p(x_{m_k}, x_{n_k}), p(y_{m_k}, y_{n_k})\}) = \epsilon$

II. $\epsilon \leq$
$s^2 \liminf\limits_{k\to 0}(max\{p(x_{m_k-1}, x_{n_k-1}), p(y_{m_k-1}, y_{n_k-1})\})$
$\leq s^2 \limsup\limits_{k\to\infty}(max\{p(x_{m_k-1}, x_{n_k-1}), p(y_{m_k-1}, y_{n_k-1})\})$
$\leq s^3\epsilon$

III. ϵ
$\leq s\liminf\limits_{k\to\infty}(max\{p(x_{m_k}, x_{n_k-1}), p(y_{m_k}, y_{n_k-1})\})$
$\leq s\limsup\limits_{k\to\infty}(max\{p(x_{m_k}, x_{n_k-1}), p(y_{m_k}, y_{n_k-1})\})$
$\leq s\epsilon$

Now $p(x_{m_k+1}, x_{n_k}) = p(F(x_{m_k}, y_{m_k}), F(x_{n_k-1}, y_{n_k-1}))$
$\leq \alpha((x_{m_k}, y_{m_k}), (x_{n_k-1}, y_{n_k-1}))$
$p(F(x_{m_k}, y_{m_k}), F(x_{n_k-1}, y_{n_k-1}))$
$\leq \psi(max\{p(x_{m_k}, x_{n_k-1}), p(y_{m_k}, y_{n_k-1})\})$ (3.1.6)

Similarly $p(y_{m_k+1}, y_{n_k}) \leq$
$\psi(max\{p(y_{m_k}, y_{n_k-1}), p(x_{m_k}, x_{n_k-1})\}$ (3.1.7)

$\therefore$ from (3.1.6) and (3.1.7) we have
$\epsilon \leq max\{p(x_{m_k+1}, x_{n_k}), p(y_{m_k+1}, y_{n_k})\} \leq$
$\psi(max\{p(x_{m_k}, x_{n_k-1}), p(y_{m_k}, y_{n_k-1})\}) \leq \psi(\epsilon)$
$< \dfrac{\epsilon}{s}$ (3.1.8)

Now, $p(x_{m_k}, x_{n_k}) \leq s(p(x_{m_k}, x_{m_k+1}) +$
$p(x_{m_k+1}, x_{n_k})) - p(x_{m_k+1}, x_{m_k+1})$
$\leq s(p(x_{m_k}, x_{m_k+1}) + p(x_{m_k+1}, x_{n_k}))$
$\leq sp(x_{m_k}, x_{m_k+1}) + \epsilon$ (by (3.1.8))

Similarly, $p(y_{m_k}, y_{n_k}) \leq sp(y_{m_k+1}, y_{m_k}) + \epsilon$
(by (3.1.8))

$\therefore \epsilon \leq max\{p(x_{m_k}, x_{n_k}), p(y_{m_k}, y_{n_k})\} \leq$
$s\,max\{p(x_{m_k+1}, x_{m_k}), p(y_{m_k+1}, y_{m_k})\} + \epsilon$

Allowing $k\to\infty$ and using (3.1.4)
$\epsilon \leq max\{p(x_{m_k}, x_{n_k}), p(y_{m_k}, y_{n_k})\} \leq \epsilon$

$\therefore \lim\limits_{k\to\infty}(max\{p(x_{m_k}, x_{n_k}), p(y_{m_k}, y_{n_k})\})$ exists and equal to ϵ

Therefore (I) follows.

Again we have $p(x_{m_k}, x_{n_k}) \leq s(p(x_{m_k}, x_{n_k-1})$ $+$
$p(x_{n_k-1}, x_{n_k}))$
$\leq s(s(p(x_{m_k}, x_{m_k-1}) + p(x_{m_k-1}, x_{n_k-1}))$ -
$p(x_{m_k-1}, x_{m_k-1})) + sp(x_{n_k-1}, x_{n_k})$
$\leq s^2 p(x_{m_k}, x_{m_k-1}) + s^2 p(x_{m_k-1}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k})$

Similarly $p(y_{m_k}, y_{n_k}) \leq s^2 p(y_{m_k}, y_{m_k-1}) +$
$s^2 p(y_{m_k-1}, y_{n_k-1})$ $+$ $sp(y_{n_k-1}, y_{n_k})$

$\therefore \epsilon \leq max\{p(x_{m_k}, x_{n_k}), p(y_{m_k}, y_{n_k})\} \leq$
$s^2 max\{p(x_{m_k}, x_{m_k-1}), p(y_{m_k}, y_{m_k-1})\} +$
$s^2 max\{p(x_{m_k-1}, x_{n_k-1}), p(y_{m_k-1}, y_{n_k-1})\} +$
$s\,max\{p(x_{n_k}, x_{n_k-1}), p(y_{n_k}, y_{n_k-1})\}$

Allowing $k\to\infty$ and using (3.1.4)

We have $\epsilon \leq s^2 \liminf\limits_{k\to\infty}$
$(max\{p(x_{m_k-1}, x_{n_k-1}), p(y_{m_k-1}, y_{n_k-1})\})$ (3.1.9)

On the other hand,
$p(x_{m_k-1}, x_{n_k-1})$
$\leq s(p(x_{m_k-1}, x_{m_k}) + p(x_{m_k}, x_{n_k-1})) - p(x_{m_k}, x_{n_k})$
$\leq s(p(x_{m_k-1}, x_{m_k})$ $+$ $p(x_{m_k}, x_{n_k-1}))$
$\leq sp(x_{m_k-1}, x_{m_k}) + s\epsilon$

Similarly $p(y_{m_k-1}, y_{n_k-1}) \leq sp(y_{m_k-1}, y_{m_k}) + s\epsilon$

$\therefore max\{p(x_{m_k-1}, x_{n_k-1}), p(y_{m_k-1}, y_{n_k-1})\} \leq$
$s\,max\{p(x_{m_k-1}, x_{m_k}), p(y_{m_k-1}, y_{m_k})\} + s\epsilon$ (3.1.10)

Allowing $k\to\infty$ and using (3.1.4)
$\limsup\limits_{k\to\infty}(max\{p(x_{m_k-1}, x_{n_k-1}), p(y_{m_k-1}, y_{n_k-1})\})$ (3.1.1)
$\leq s\epsilon$

$\therefore$ From (3.1.9) and (3.1.11)
$\epsilon \leq$
$s^2 \liminf\limits_{k\to 0}(max\{p(x_{m_k-1}, x_{n_k-1}), p(y_{m_k-1}, y_{n_k-1})\})$
$\leq s^2 \limsup\limits_{k\to\infty}(max\{p(x_{m_k-1}, x_{n_k-1}), p(y_{m_k-1}, y_{n_k-1})\})$
$\leq s^3\epsilon$

Therefore (II) follows.

Finally we have ,
$p(x_{m_k}, x_{n_k}) \leq s(p(x_{m_k}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}))$

$\leq s\epsilon + sp\,(x_{n_k-1}, x_{n_k}))$

Similarly

$p\,(y_{m_k}, y_{n_k}) \leq s(p\,(y_{m_k}, y_{n_k-1}) + p\,(y_{n_k-1}, y_{n_k}))$

$\leq s\epsilon + sp\,(y_{n_k-1}, y_{n_k}))$

$\therefore \epsilon \leq$

$max\,\{p\,(x_{m_k}, x_{n_k}), p\,(y_{m_k}, y_{n_k})\}$

$\leq smax\,\{p\,(x_{m_k}, x_{n_k-1}), p\,(y_{m_k}, y_{n_k-1})\} + smax\,\{p\,(x_{n_k-1}, x_{n_k}), p\,(y_{n_k-1}, y_{n_k})\}$

$\leq s\epsilon + s\,max\,\{p\,(x_{n_k-1}, x_{n_k}), p\,(y_{n_k-1}, y_{n_k})\}$

Allowing $k \to \infty$ and using (3.1.4)

ϵ

$\leq s \liminf_{k\to\infty}(max\,\{p\,(x_{m_k}, x_{n_k-1}), p\,(y_{m_k}, y_{n_k-1})\})$

$\leq s \limsup_{k\to\infty}(max\,\{p\,(x_{m_k}, x_{n_k-1}), p\,(y_{m_k}, y_{n_k-1})\})$

$\leq s\epsilon$

Therefore (III) follows.

Now from (3.1.1) $p(x_{m_k}, x_{n_k}) =$

$p\,(F(x_{m_k-1}, y_{m_k-1}), F(x_{n_k-1}, y_{n_k-1}))$

$\leq \alpha((x_{m_k-1}, y_{m_k-1}), (x_{n_k-1}, y_{n_k-1}))$

$p\,(F(x_{m_k-1}, y_{m_k-1}), F(x_{n_k-1}, y_{n_k-1}))$

$\leq \psi(max\{p(x_{m_k-1}, x_{n_k-1}), p(y_{m_k-1}, y_{n_k-1})\})$ (3.1.12)

Similarly

$p\,(y_{m_k}, y_{n_k}) = p\,(F(y_{m_k-1}, x_{m_k-1}), F(y_{n_k-1}, x_{n_k-1}))$

$\leq p\,(F(y_{m_k-1}, x_{m_k-1}), F(y_{n_k-1}, x_{n_k-1}))$

$\alpha((y_{m_k-1}, x_{m_k-1}), (y_{n_k-1}, x_{n_k-1}))$

$\leq \psi(max\{p(y_{m_k-1}, y_{n_k-1}), p(x_{m_k-1}, x_{n_k-1})\}$ (3.1.13)

$\therefore$ from (3.1.10) and (3.1.11) we have

$\epsilon \leq max\{p\,(x_{m_k}, x_{n_k}), p\,(y_{m_k}, y_{n_k})\}$

$\leq \psi(max\{p(x_{m_k-1}, x_{n_k-1}), p(y_{m_k-1}, y_{n_k-1})$

$= \psi(t_k) \leq \frac{1}{s} t_k$ (3.1.14)

where $t_k = max\{p(x_{m_k-1}, x_{n_k-1}), p(y_{m_k-1}, y_{n_k-1})$

$\therefore s\epsilon < t_k$ for every k

But by (II), $\liminf_{k\to\infty}\,(t_k) \leq \limsup_{k\to\infty}\,(t_k) \leq s\epsilon$

Hence $s\epsilon \leq \liminf_{k\to\infty}\,(t_k) \leq \limsup_{k\to\infty}\,(t_k) \leq s\epsilon$

$\therefore \lim_{k\to\infty}\,(t_k) = s\epsilon \Rightarrow \lim_{k\to\infty}\,(\frac{t_k}{s}) = \epsilon$

Hence from (3.1.14), $\epsilon \leq \lim_{k\to\infty}\,\psi(t_k) \leq \lim_{k\to\infty}\,(\frac{t_k}{s}) = \epsilon$

$\therefore \lim_{k\to\infty}\,\psi(t_k) = \epsilon$

From (3.1.14) again, $\epsilon \leq \psi(t_k) < \frac{t_k}{s} < t_k$

$\therefore \epsilon = \lim_{k\to\infty}\,\psi(t_k) = \lim_{t_k\to(s\epsilon)^+}\,\psi(t_k) < \frac{s\epsilon}{s} = \epsilon,$

is a contradiction.

$\therefore$ sequences $\{x_n\}, \{y_n\}$ are Cauchy sequences.

$\therefore$ by(3.1.4), $\lim_{m,n\to\infty}\,p\,(x_m, x_n) = 0$

and $\lim_{m,n\to\infty}\,p\,(y_m, y_n) = 0$

Let sequence $\{x_n\} \to x$ and $\{y_n\} \to y$ as $n \to \infty$

$\therefore \lim_{n\to\infty}\,p\,(x_n, x) = 0 = p\,(x, x)$

and $\lim_{n\to\infty}\,p\,(y_n, y) = 0 = p\,(y, y)$

We have $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$

Now $\lim_{n\to\infty}\,x_{n+1} = \lim_{n\to\infty}\,F\,(x_n, y_n) = F(x, y)$

and also $\lim_{n\to\infty}\,x_{n+1} = x$

$\therefore$ By lemma 2.8 and (3.1.4), we have $x = F(x, y)$

Similarly $y = \lim_{n\to\infty}\,y_{n+1} = \lim_{n\to\infty} F\,(y_n, x_n) = F(y, x)$

$\therefore$ x and y are coupled fixed points of F in X.

Now we state and prove our second main result:

Theorem 3.2.

Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s > 1$. Let $F : X \times X \to X$ be a mapping having the mixed monotone property of X. Suppose $\psi \in \Psi_s$ and $\alpha : X^2 \times X^2 \to [0, \infty)$ such that for $x, y, u, v \in X$, the following holds :

$\alpha((x, y), (u, v)) \quad p(F(x, y), F(u, v)) \leq \psi(max\{p(x, u), p(y, v)\}) \;\; \forall\, x \geq u, y \leq v$ (3.2.1)

Suppose that

(i) F is α - admissible.

(ii) there exist $x_0, y_0 \in X$ such that

$\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1,$

$\alpha((y_0, x_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1.$

and

$x_0 \leq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$

(iii)If $\{x_n\}$ is a non-decreasing sequence and $\{y_n\}$ is a non-increasing sequence, then $\lim_{n\to\infty}\,x_n = x \Rightarrow x_n \leq x$ and $\lim_{n\to\infty}\,y_n = y \Rightarrow y_n \geq y \,\forall\, n \in N$

(iv) Further If $\alpha((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1$ and $\alpha((y_n, x_n), (y_{n+1}, x_{n+1})) \geq 1 \quad \forall\, n$, then $\alpha((x_n, y_n), (x, y)) \geq 1$ and $\alpha((y_n, x_n), (y, x)) \geq 1$

Then F has a coupled fixed point, that is~ $\exists x, y \in X$ such that $F(x,y)=x$ and $F(y,x)=y$.

Proof: Using (ii), define the sequences $\{x_n\}$ and $\{y_n\}$ as in Theorem 3.1. Then it can be shown, as in Theorem 3.1, sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

Since sequence $\{x_n\}$ is non-decreasing and sequence $\{y_n\}$ is non-increasing, by (iii), $\lim_{n\to\infty} x_n = x \Rightarrow x_n \le x$ and $\lim_{n\to\infty} y_n = y \Rightarrow y_n \ge y$

and also from (3.1.2) and (3.1.3) of Theorem 3.1

$\alpha((x_n,y_n),(x,y)) \ge 1$ and $\alpha((y_n,x_n),(y,x)) \ge 1$

$$\therefore\ p(x_{n+1},F(x,y)) = p(F(x_n,y_n),F(x,y)) \le \alpha((x_n,y_n),(x,y))\, p(F(x_n,y_n),F(x,y)) \le \psi(max\{(p(x_n,x),p(y_n,y))\})$$

$\forall\ x_n \le x$ and $y_n \ge y$

Similarly

$$p(y_{n+1},F(y,x)) \le \psi(max\{(p(y_n,y),p(x_n,x))\})$$

$\forall\ x_n \le x$ and $y_n \ge y$

$$\therefore \max\{p(x_{n+1},F(x,y)),p(y_{n+1},F(y,x))\} \le \psi(max\{(p(x_n,x),p(y_n,y))\})$$

$$\therefore\ \lim_{n\to\infty} \max\{p(x_{n+1},F(x,y)),p(y_{n+1},F(y,x))\} \le \lim_{n\to\infty} \psi(max\{(p(x_n,x),p(y_n,y))\})$$

$\le \lim_{n\to\infty} max\{(p(x_n,x),p(y_n,y))\}) = 0$ (from (3.1.5) of theorem 3.1)

$\therefore \lim_{n\to\infty} p(x_{n+1},F(x,y)) = 0$ and $\lim_{n\to\infty} p(y_{n+1},F(y,x)) = 0$

$\Rightarrow \lim_{n\to\infty} x_{n+1} = F(x,y))$ and $\lim_{n\to\infty} y_{n+1} = F(y,x))$

Since sequence $\{x_n\}$ and sequence $\{y_n\}$ are Cauchy sequences and $\lim_{n\to\infty} x_n = x$;

$\lim_{n\to\infty} y_n = y \Rightarrow \lim_{n\to\infty} x_{n+1} = x$ and $\lim_{n\to\infty} y_{n+1} = y$

Further $p(x,F(x,y))$

$\le\ s(p(x,x_{n+1})\ +\ p(x_{n+1},F(x,y)))$ - $p(x_{n+1},x_{n+1})$

$\therefore\ p(x,F(x,y)) = 0$ and similarly $p(y,F(y,x)) = 0$

Hence by lemma 2.9 , $x = F(x,y)$ and $y = F(y,x)$

$\therefore$ F has coupled fixed point in X .

Now we state and prove our third main result.

In this result, we obtain a sufficient condition which assumes the uniqueness of coupled fixed point.

Theorem 3.3. Let $(X, \le, p)$ be a complete partially ordered partial b - metric space with coefficient S > 1. Let $F: X\times X \to X$ be a mapping having the mixed monotone property of X. Suppose $\psi \in \Psi_s$ and $\alpha: X^2 \times X^2 \to [0,\infty)$ such that for $x,y,u,v \in X$, the following holds :

$$\alpha((x,y),(u,v))\ p(F(x,y),F(u,v)) \le \psi(max\{p(x,u),p(y,v)\}) \quad (3.3.1)$$

$\forall\, x \ge u\, and\, y \le v$. Suppose that

(i) F is α - admissible.

(ii) there exist $x_0, y_0 \in X$ such that

$\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))) \ge 1$,

$\alpha((y_0,x_0),(F(y_0,x_0),F(x_0,y_0))) \ge 1$ and

$x_0 \le F(x_0,y_0)$ and $y_0 \ge F(y_0,x_0)$.

(iii) F is continuous Suppose for every (x,y) and (w,t) in $X\times X$, there exists (u,v) such that $\alpha((x,y),(u,v)) \ge 1$ and $\alpha((w,t),(u,v)) \ge 1$.

Then F has a unique coupled fixed point.

Proof: We have by theorem 3.1 the set of coupled fixed points is non-empty. Suppose (x,y) and (w,t) are coupled points of the mapping $F: X\times X \to X$, that is $x = F(x,y)$, $y = F(y,x)$ and $w = F(w,t)$, $t = F(t,w)$. By assumption, there exists (u,v) in $X\times X$ such that (u,v) is comparable to (x,y) and (w,t). Let $u = u_0$ and $v = v_0$.

Write $u_1 = F(u_0,v_0)$ and $v_1 = F(v_0,u_0)$. Thus inductively we can define sequences $\{u_n\}$ and $\{v_n\}$ as $u_{n+1} = F(u_n,v_n)$ and $v_{n+1} = F(v_n,u_n)$ Since (u,v) is comparable to (x,y), therefore

$x \le u = u_0 \Rightarrow F(x,y) \le F(u_0,y) \le F(u_0,v_0)$ $(\because y \ge v_0)$

$\Rightarrow x \le F(u_0,v_0) = u_1$ and similarly $y \ge v_1$

$\Rightarrow\ x \le u_1$ and $y \ge v_1$ and by induction $x \le u_n$ and $y \ge v_n$ $\forall n \ge 1$.

Since $u = u_0$ and $v = v_0$, we get $\alpha((x,y),(u_0,v_0)) \ge 1$ $\Rightarrow$

$\alpha((F(x,y),F(y,x)),(F(u_0,v_0),F(v_0,u_0))) \ge 1 \Rightarrow$

$\alpha((x,y),(u_1,v_1)) \ge 1$

$\therefore$ By mathematical induction, we obtain

$$\alpha((x,y),(u_n,v_n)) \ge 1\ \ \forall n \ge 1 \quad (3.3.2)$$

Similarly $\alpha((y,x),(v_n,u_n)) \ge 1\ \ \forall n \ge 1$ (3.3.3)

$\therefore$ from $(3.3.1),(3.3.2),(3.3.3)$ we have

$$p\ (x,u_{n+1}) = p\ (F(x,y),F(u_n,v_n)) \le\ \alpha((x,y),(u_n,v_n))\, p\ (F(x,y),F(u_n,v_n)) \le\ \psi(max\{(x,u_n),(y,v_n)\}) \quad (3.3.4)$$

Similarly

$$p\ (y,v_{n+1}) \le \psi(max\{(y,v_n),(x,u_n)\}) \quad (3.3.5)$$

$\therefore$ from $(3.3.4),(3.3.5)$ we have

$max\{p\ (x,u_{n+1}), p\ (y,v_{n+1})\}$
$\leq \psi(max\{(x,u_n),(y,v_n)\}) \Rightarrow \ \mu_{n+1} \ \leq \ \psi(\mu_n)$
where μ_{n+1} = $max\ \{p\ (x,u_{n+1}), p\ (y,v_{n+1})\}$
$\therefore\ \mu_{n+1} \ \leq \ \psi(\mu_n) \ \leq \ \psi^2(\mu_{n-1}) \ \leq ... \ \leq \ \psi^{n+1}(\mu_0)$
where μ_0 = $max\ \{p\ (x,u_0), p\ (y,v_0)\}$
$\therefore$ sequence $\{\mu_n\}$ is decreasing and $\lim_{n\to\infty}$ $\mu_{n+1} \leq \lim_{n\to\infty} \psi^{n+1}(\mu_0) \ = 0$ (by (i) lemma 3.0)
$\therefore$ sequence $\{\mu_n\}$ is convergent and converges to 0
$\therefore \lim_{n\to\infty} \ \mu_{n+1} = 0 \ \Rightarrow \lim_{n\to\infty} max\ \{p\ (x,u_{n+1}), p\ (y,v_{n+1})\}$ $= 0$ (3.3.6)
$\therefore \lim_{n\to\infty} \{p\ (x,u_{n+1})\} = 0$ and $\lim_{n\to\infty} \ \{p\ (y,v_{n+1})\} = 0$
Similarly
$\lim_{n\to\infty} \{p\ (w,u_{n+1})\} = 0$ and $\lim_{n\to\infty} \{p\ (t,v_{n+1})\} = 0$
Now
$p\ (x,w)$
$\leq s(p\ (x,u_{n+1}) + p\ (u_{n+1},w)) - p\ (u_{n+1},u_{n+1})$
$\leq sp\ (x,u_{n+1}) + sp\ (u_{n+1},w)$
$\therefore p(x,w)$
$\leq \ s\lim_{n\to\infty} \ p\ (x,u_{n+1}) \ + \ s\lim_{n\to\infty} \ p\ (u_{n+1},w) = 0$
$\Rightarrow \ p\ (x,w) = 0$
Similarly $p\ (y,t) = 0$
Hence by lemma 2.9, $x = w$ and $y = t$
$\therefore$ F has unique coupled fixed point.

The following theorem which is the fourth main result can be established on the lines of the proof of theorems 3.1 and 3.2

Theorem 3.4. Let $(X,\leq,p)$ be a complete partially ordered partial b - metric space with $ s > 1$. Let $F: X\times X \to X$ be a mapping having the mixed monotone property of X. Suppose $\psi \in \Psi_s$ and $\alpha: X^2\times X^2 \to [0,\infty)$ such that for $x,y,u,v \in X$, the following holds :

$\alpha((x,y),(u,v))$ $p(F(x,y),F(u,v))$ $\leq$ $\psi(max\{p(x,u), p(y,v)\})$ (3.4.1)

$\forall\, x \geq u\, and\ y \leq v$. Suppose that

(i) F is α - admissible.

(ii) there exist $x_0, y_0 \in X$ such that $\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))) \geq 1$, $\alpha((y_0,x_0),(F(y_0,x_0),F(x_0,y_0))) \geq 1$.and $x_0 \geq F(x_0,y_0)$ and $y_0 \leq F(y_0,x_0)$.

(iii) F is continuous (Or)

(iv)If sequence $\{x_n\}$ is non-increasing and sequence $\{y_n\}$ is non-decreasing then $\lim_{n\to\infty} \ x_n = x \ \Rightarrow x_n \geq x$ and $\lim_{n\to\infty} y_n = y \ \Rightarrow y_n \leq y\, \forall\, n \in N$;

Further If ~ $\alpha((x_n,y_n),(x_{n+1},y_{n+1})) \geq 1$ and $\alpha((y_n,x_n),(y_{n+1},x_{n+1})) \geq 1$ $\forall n$, then $\alpha((x_n,y_n),(x,y)) \geq 1$ and $\alpha((y_n,x_n),(y,x)) \geq 1$

Then F has a coupled fixed point, that is $\exists x,y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$.

Proof: follows as in the theorems 3.1 and 3.2

Now we give an example in support of theorem 3.1

3.5 Example: Let $X = [0,1]$ with usual ordering. Define $p(x,y) = (max\{x,y\})^2 \ \ \forall\, x,y \in X$. Clearly, $(X,\leq,p)$ is a partially ordered partial b - metric space with coefficient $s = 2$

Define $F: X\times X \to X$ by

$$F(x,y) = \begin{cases} \frac{1}{2}x(1-y) \text{ if } \forall\, x,y \in X \end{cases}$$

Define $\alpha: X^2\times X^2 \to [0,\infty)$ by

$$\alpha((x,y),(u,v))) = \begin{cases} 1 \text{ if } \ x\geq y, u \geq v \\ 0 \text{otherwise} \end{cases}$$

Define $\psi(t) = \frac{t}{3} \ \Rightarrow \psi(t) < \frac{t}{2} \Rightarrow \ \psi(t) < \frac{t}{s}$

Without loss of generality, we may assume that $0 \leq y \leq x \leq 1$

$\therefore\ p(x,y) = x^2$,

$p(F(x,y),F(u,v))$ = (max$\{F(x,y),F(u,v)\})^2$

$= max(\{\frac{1}{2}x(1-y)\}^2, \{\frac{1}{2}u(1-v)\}^2)$

$p(x,u) = max(x^2,u^2),\ p(y,v) = max(y^2,v^2)$,

$\therefore$ $\psi(max\{p(x,u), p(y,v)\})$

$= \psi(max\{x^2,u^2\}, max\{y^2,v^2\}) = \frac{1}{3}(max\{x^2,u^2\})$

(since $x \geq y$ and $u \geq v$)

Now $\alpha((x,y),(u,v))$ $p(F(x,y),F(u,v)) = p(F(x,y),F(u,v))$

$= max(\{\frac{1}{2}x(1-y)\}^2, \{\frac{1}{2}u(1-v)\}^2)$

and $\psi(max\{p(x,u), p(y,v)\})$ = $\frac{1}{3}(max\{x^2,u^2\})$

Now we show that

$\alpha((x,y),(u,v)))$ $p(F(x,y),F(u,v))$ $\leq$ $\psi(max\{p(x,u), p(y,v)\})$ (3.5.1)

That is, we show that $max(\{\frac{1}{2}x(1-y)\}^2, \{\frac{1}{2}u(1-v)\}^2)$

$$\leq \frac{1}{3}(max\{x^2, u^2\}) \quad (3.5.2)$$

The following are the cases

Case(1): Let $\{\frac{1}{2}x(1-y)\}^2 \geq \{\frac{1}{2}u(1-v)\}^2$ and $\frac{1}{3}x^2 \geq \frac{1}{3}u^2$

$\because \{\frac{1}{2}x(1-y)\}^2 \leq \frac{1}{3}x^2 \Rightarrow$ (3.5.2) holds $\Rightarrow$ (3.5.1) holds

Case(2): Let $\{\frac{1}{2}x(1-y)\}^2 \geq \{\frac{1}{2}u(1-v)\}^2$ and $\frac{1}{3}u^2 \geq \frac{1}{3}x^2$

$\because \{\frac{1}{2}x(1-y)\}^2 \leq \frac{1}{3}x^2 \leq \frac{1}{3}u^2 \Rightarrow$ (3.5.2) holds $\Rightarrow$ (3.10.1) holds

Case(3): Let $\{\frac{1}{2}x(1-y)\}^2 \leq \{\frac{1}{2}u(1-v)\}^2$ and $\frac{1}{3}x^2 \geq \frac{1}{3}u^2$

$\because \{\frac{1}{2}u(1-v)\}^2 \leq \frac{1}{3}u^2 \leq \frac{1}{3}x^2 \Rightarrow$ (3.5.2) holds $\Rightarrow$ (3.5.1) holds

Case(4): Let $\{\frac{1}{2}x(1-y)\}^2 \leq \{\frac{1}{2}u(1-v)\}^2$ and $\frac{1}{3}u^2 \geq \frac{1}{3}x^2$

$\because \{\frac{1}{2}u(1-v)\}^2 \leq \frac{1}{3}u^2 \Rightarrow$ (3.5.2) holds $\Rightarrow$ (3.5.1) holds

Hence $\alpha((x,y),(u,v))) \quad p(F(x,y),F(u,v)) \leq \psi(max\{p(x,u),p(y,v)\})$

(i) Let $x^{'} = F(x,y) = \frac{1}{2}x(1-y)$ and $y^{'} = F(y,x) = \frac{1}{2}y(1-x)$

Further let $u^{'} = F(u,v) = \frac{1}{2}u(1-v)$ and $v^{'} = F(v,u) = \frac{1}{2}v(1-u)$

Since $x^{'}, y^{'}, u^{'}, v^{'} \in X$ and $x^{'} \geq y^{'}$, $u^{'} \geq v^{'}$

$\therefore \alpha((x^{'},y^{'}),(u^{'},v^{'})) \geq 1 \Rightarrow$ F is α - admissible.

(ii) Taking $0 = x_0, 0 = y_0 \Rightarrow F(x_0,y_0) = 0, F(y_0,x_0) = 0$

then $\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))) = 1$ and $\alpha((y_0,x_0),(F(y_0,x_0),F(x_0,y_0))) = 1$

(iii) F is continuous.

$\therefore$ F satisfies the hypothesis of theorem 2.6

Now $F(0,0) = 0$ and $F(0,0) = 0$

Hence (0,0) is a coupled fixed point of F in X +

Note It can be shown that this example also supports Theorem 3.2 .

Open Problem: Are the theorems 3.1, 3.2, 3.3 and 3.4 true for partial b - metric spaces with coefficient $s \geq 1$ when ψ is defined independent of *s* ?

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