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STATISTICAL DESIGN OF EWMA CHART FOR MA(Q) BASED ON ARL

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Abstract: **The objective of this paper is to show an explicit formulas of the Average Run Length (ARL) for Exponentially Weighted Moving Average (EWMA) chart when observations are described by Moving Average order q (MA(q)) processes with exponential white noise. The ARL is a traditional measurement of control chart's performance, the expected number of observations taken from an in-control process until the control chart falsely signals out-of-control is denoted by ARL_0. An ARL_0 will be regarded as acceptable if it is large enough to keep the level of false alarms at an acceptable level. A second common characteristic is the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control is denoted by ARL_1. In particular, the explicit analytical formulas for evaluating ARL_0 and ARL_1 be able to get a set of optimal parameters which depend on a smoothing parameter (λ) and width of control limit (b) for designing EWMA chart with minimum of ARL_1.**

Keywords: **Exponentially Weighted Moving Average chart, Average Run Length, Integral Equations.**

I. INTRODUCTION

Statistical Process Control (SPC) play a vital role in monitoring, detecting changes in a processes, and uses for measuring, controlling and improving quality in areas such as industrial and manufacturing, finance and economics, computer sciences, epidemiology, public health surveillance and in other areas of applications (see Frisen [1]; Noorossana et al [2]; Mazalov and Zhuralav[3]). The control chart is an important statistical technique that is used to monitor the quality of a process. All popular charts such as Shewhart, Exponentially Weighted Moving Average (EWMA) and Cumulative Sum charts have been developed for detecting changes in a process means. The Shewhart chart was introduced by Shewhart [4] which is widely used in many applications as the main tool for detecting large changes in a process mean. However, the Shewhart chart has been found to be inadequate for detecting small shifts in process means. In the past few decades, the Cumulative Sum (CUSUM) and the Exponentially Weighted Moving Average (EWMA) charts have been proposed as good alternatives to the Shewhart chart for detecting small shifts. The Exponentially Weighted Moving Average (EWMA) chart is very common and effective procedure which was first introduced by Robert [5]. It is a very flexible and effective chart for detecting small changes and has the advantage of showing robustness to non-normality (Borror *et al.*[6]). Later, Nong et al. [7] implemented EWMA chart for monitoring the events intensity for intrusion network systems. Han [8] employed EWMA and CUSUM charts in economics and finance to detected turning point in the IBM's stock. In addition, EWMA chart were applied in lifetime observations see e.g Zhang and Chen[9].

The characteristic of control chart is Average Run Length (ARL) which is the average number of samples taken before an action signal is given. The ARL should be sufficiently large while the process is still in-control is denoted by ARL_0 and the Average Delay time which is mean delay of true alarm times. It should be small when the process goes out-of-control is denoted by ARL_1. Many methods for evaluating the ARL_0 and ARL_1 for control charts have been studied in the literature. A simple approach that is often used to test other methods is Monte Carlo (MC) simulation. MC is simple to program and is convenient for controlling and testing accuracy of analytical approximations. However, MC is usually based on a large number of sample trajectories so it is very time consuming. Markov Chain Approach (MCA) is considered as a popular technique. It is based on approximation of Markov Chains by using matrix inversions. Although there are at present no theoretical results on accuracy of this procedure, the results have been tested by direct comparison with MC simulations. Integral Equation (IE) is the most advanced method currently available. However, the results for ARL_0 and ARL_1 usually cannot be obtained analytically and intensive programming or specialized software is required to obtain numerical results. Recently, explicit formulas for evaluation ARL have been presented. Sukparungsee and Novikov [10] have used the Martingale approach to derive approximate analytical formulas for ARL and AD in the case of Gaussian distribution and some Non-Gaussian distribution. Later, Areepong and Novikov [11] derived the explicit formulas of Average Run Length (ARL) and Average Delay (AD) for EWMA control chart for the case of Exponential distribution.

Traditional SPC technique is based on the fundamental assumption that random data are independent identically distributed. However, this is not always an assumption of practical interest in applications, e.g., in chemical industry, were random observed data are serially dependent, so its need to be monitored by appropriate control charts. Mastrangelo, C.M. and Montgomery, D.C. [12] have been evaluated the performance of EWMA control charts for serially-correlated process based on Monte Carlo simulation technique. Vanbrackle, L. N. and Reynold, M.R. [13] were estimated the ARL by using an Integral Equation and Markov Chain Approach to evaluate EWMA and CUSUM control charts in case of AR(1) process with additional random error. Recently, Mititelu et al. [14] presented the explicit formulas for ARL by Fredholm Integral Equation for one-sided EWMA control chart with Laplace distribution and CUSUM control chart with Hyperexponential distribution. Later, Busaba et al. [15] was analyzed the explicit formulas of ARL for CUSUM control chart, its corresponding in the case of a Stationary First Order Autoregressive: AR(1) process with Exponential white noise. Consequently, the aim of paper is to present the explicit formulas of Average Run Length (ARL) of EWMA control chart for Moving Average: MA (q) process with exponential white noise. Using the explicit formulas we have been able to provide the tables for the smoothing parameter (λ) and width of control limit (b) for designing EWMA chart with minimum of ARL_1.

II. THE AVERAGE RUN LENGTH (ARL) FOR EWMA CHART OF MA(Q) PROCESSE WITH EXPONENTIAL WHITE NOISE

In this paper we consider SPC charts under the assumption that sequential observations $\xi_1, \xi_2, \ldots,$ are independent random variables with a distribution function $F(x,\alpha)$, the parameter $\alpha = \alpha_0$ before a change-point time $\theta \le \infty$ ("in-control" state; $\theta = \infty$ means that there are no change at all) and $\alpha > \alpha_0$ after the change-point time θ ("out-of-control" state).

All popular charts like Shewhart, Cumulative Sum (CUSUM) and EWMA charts are based on use of stopping times τ. The typical condition on choice of the stopping times τ is the following:

$$E_\infty(\tau) = T, \tag{1}$$

where T is given (usually large), and $E_\infty(.)$ denote that the expectation under distribution $F(x,\alpha_0)$ (in-control) that the change-point occurs at point θ (where $\theta \le \infty$). In literature on quality control the quantity $E_\infty(\tau)$ is called as Average Run Length (ARL) for in-control process of the algorithm. Then, by definition, ARL_0 = $E_\infty(\tau)$ and the typical practical constraint is $ARL_0 = T$.

Another typical constraint consists in minimizing the quantity

$$Q(\alpha) = E_\theta(\tau - \theta + 1 | \tau \ge \theta), \tag{2}$$

where $E_\theta(.)$ is the expectation under distribution $F(x,\alpha)$ (out-of-control) and α is the value of parameter after the change-point. We restrict on the special case, usually $\theta = 1$. The quantity $E_1(\tau)$ is called as Average Run Length for out-of-control process (ARL_1) and one could expect that a sequential chart has a near optimal performance if its ARL_1 is close to a minimal value.

The EWMA statistics based on MA(q) process is defined by the following recursion:

$$X_t = (1-\lambda)X_{t-1} + \lambda Z_t, \quad t = 1,2,\ldots \tag{3}$$

where X_t is the EWMA statistics, Z_t is a sequence of MA(q) process and the initial value is a constant ($Z_0 = u$) and $\lambda \in (0,1)$ is smoothing parameter.

The corresponding stopping time for expressions (3) define as

$$\tau = \inf\{t > 0; X_t > b\}, \quad X_0 = u, \; b > x. \tag{4}$$

where b denote control limit.

The general Moving Average process, denoted by MA(q) process can be written as:

$$Z_t = \xi_t - \theta_1\xi_{t-1} - \theta_2\xi_{t-2} - \ldots - \theta_q\xi_{t-q}$$

where ζ_t is to be a white noise processes assumed with Exponential distribution. An moving average coefficient $-1 \le \theta_i \le 1$.

III. SOLUTION FOR EVALUATING ARL0 AND ARL1 OF EWMA PROCEDURE

In this section we present the explicit formulas for ARL which is submitted in Petcharat [16]. We assume that, the process initially in-control $X_0 = u$. The integral equation defines in $L(u)$ as follow;

$$L(u) = 1 + \frac{1}{\lambda}\int_0^b L(y)\, f\left(\frac{y-(1-\lambda)u}{\lambda} + \left(\xi_t + \theta_1\xi_{t-1} + \theta_2\xi_{t-2} + \ldots + \theta_q\xi_{t-q}\right)\right) d(y)$$

$$L(u) = 1 + \frac{1}{\lambda\alpha}\int_0^b L(y)\, e^{\frac{y}{\lambda\alpha}} e^{\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\xi_t + \theta_1\xi_{t-1} + \theta_2\xi_{t-2} + \ldots + \theta_q\xi_{t-q}\right)}{\alpha}} d(y) \tag{5}$$

Let $C(u) = \exp\left[\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\xi_t + \theta_1\xi_{t-1} + \theta_2\xi_{t-2} + \ldots + \theta_q\xi_{t-q}\right)}{\alpha}\right]$

then the function $L(u)$ in (5) can be written as

$$L(u) = 1 + \frac{C(u)}{\lambda\alpha}\int_0^b L(y)\, e^{\frac{1}{\lambda\alpha}} d(y), \quad 0 \le u \le b. \tag{6}$$

We used the second kind Fredholm integral equation to derive the ARL for MA(q) process. Now, We obtain the explicit formula for ARL_0 as follows:

$$ARL_0 = 1 - \frac{\lambda \exp\left(\frac{(1-\lambda)u}{\lambda\alpha_0}\right)\exp\left(-\frac{b}{\lambda\alpha_0}\right) - 1}{\lambda \exp\left(\frac{\theta_1\xi_{t-1} + \theta_2\xi_{t-2} + \ldots + \theta_q\xi_{t-q}}{\alpha_0}\right) + \exp\left(-\frac{b}{\alpha_0}\right) - 1} \tag{7}$$

$$ARL_1 = 1 - \frac{\lambda \exp\left(\frac{(1-\lambda)u}{\lambda\alpha_1}\right)\exp\left(-\frac{b}{\lambda\alpha_1}\right)-1}{\lambda \exp\left(\frac{\theta_1\xi_{t-1}+\theta_2\xi_{t-2}+...+\theta_q\xi_{t-q}}{\alpha_1}\right)+\exp\left(-\frac{b}{\alpha_1}\right)-1} \qquad (8)$$

Using the explicit formulas, we have been able to provide the tables for the optimal smoothing parameter (λ) and width of control limit (b) for designing EWMA chart with minimum of ARL_1. We first describe a procedure for obtaining optimal designs for EWMA chart. The criterion used for choosing optimal values for is smoothing parameter (λ) and width of control limit (b) for designing EWMA chart with minimum of ARL_1 for a given in-control parameter value α_0 =1, ARL_0 = T and a given out-of-control parameter value ($\alpha = \alpha_1$). We compute optimal (λ, b) values for T = 370 and 500 and magnitudes of change. Table of the optimal parameters values are shown in Table 3-4.

The numerical procedure for obtaining optimal parameters for EWMA designs

1. Select an acceptable in-control value of ARL and decide on the change parameter value (α_1) for an out-of-control state.

2. For given α_0 and T, find optimal values of λ and b to minimize the ARL_1 values (ARL^*_1) given by equation 8 subject to the constraint that ARL_0 = T in Equation 7, i.e. λ and b are solutions of the optimality problem.

IV. NUMERICAL RESULT

In this section, we present explicit formulas of ARL for EWMA chart when observations are moving average order q process with exponential white noise and compare to the ARL from approximated the Gauss-Legendre numerical scheme for integral equation(IE) with $m = 500$. First, define the relative error as $Diff(\%) = \frac{|Explicit - IE|}{Explicit} \times 100,$ we used Equation (7) and (8) to evaluated ARL. In Table 1 and 2, we compare the solution of explicit formula (Explicit) against numerical approximation (IE) for EWMA chart when $\alpha = 1$ in the case MA(2) and MA(3) respectively, which they are in good agreement. Notice that, the percentage of relative errors is small difference with $m = 500$ nodes. In Table 3 and 4, we use (5) and (6) to show the ARL_0 and ARL_1 results of EWMA chart for any shift size in mean δ . We use MA(2) process with parameter $\lambda = 0.05$ and given ARL_0=370, 500 with $\theta_1 = 0.1$, $\theta_2 = 0.2$. In the case of MA(3) process, we use $\lambda = 0.05$ and given ARL_0=370, 500 with $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\theta_3 = 0.3$. Obviously, the results from suggested formulas are very close to approximation IE. Note that, calculations with explicit formula from Equation (5) and (6) is simple and very fast to calculate which the computational times takes less than 1 second. The numerical results in terms of optimal width of the smoothing parameter (λ), optimal width of control limit (b) and minimum ARL_1 (ARL^*_1) for ARL =370 and 500 are shown in Table 3 and 4. For example, in the case MA(2), if we want to detect a parameter change from $\alpha = 1$ to $\alpha = 1.05$ and the ARL value is 370 then the optimality procedure given above will give optimal parameter values λ = 0.168 and H = 0.257. On substituting the values for α , λ and b into Equation 6 we obtain ARL^*_1 value = 25.146. As shown in Table 1 and 4 the use of the suggested explicit formulas for ARL_0 and $ARL_1$for EWMA chart can greatly reduce the computation times, and are useful to practitioners especially finding optimal parameters of EWMA chart.

TABLE I. COMPARISON OF ARL VALUES COMPUTED USING EXPLICIT FORMULAR (EXPLICIT) AGAINST NUMERICAL APPROXIMATION (IE) FOR MA(2) WHEN $b = 0.069597$ FOR ARL_0 =370 AND $b = 0.06966972$ FOR ARL_0 =500 WITH $\theta_1 = 0.1$ AND $\theta_2 = 0.2$

δ (shift size)	$ARL_0 = 370$		*Diff* %	$ARL_0 = 500$		*Diff* %
	Explicit	IE (CPU Time: second)		Explicit	IE (CPU Time: second)	
0.00	370.952 (0.14)	370.952 (2.901)	0.005	500.03 (0.14)	500.033 (26.99)	0.001
0.01	136.804 (0.14)	136.224 (4.307)	0.424	151.17 (0.14)	151.597 (30.828)	0.282
0.03	60.434 (0.14)	60.254 (8.113)	0.298	63.076 (0.14)	63.357 (34.681)	0.445
0.05	38.783 (0.14)	38.593 (11.826)	0.490	39.850 (0.14)	39.635 (38.503)	0.540
0.07	28.564 (0.14)	28.326 (15.679)	0.833	29.137 (0.14)	29.288 (42.294)	0.518
0.10	20.494 (0.14)	20.382 (19.486)	0.547	20.785 (0.14)	20.634 (46.767)	0.726
0.20	10.646 (0.14)	10.611 (23.214)	0.329	10.722 (0.14)	10.678 (53.542)	0.410

TABLE II. COMPARISON OF ARL VALUES COMPUTED USING EXPLICIT FORMULAR (EXPLICIT) AGAINST NUMERICAL APPROXIMATION (IE) FOR MA(2) WHEN $b = 0.095897$ FOR ARL_0 =370 AND $b = 0.0952842$ FOR ARL_0 =500 WITH $\theta_1 = 0.1$, $\theta_2 = 0.2$ AND $\theta_3 = 0.3$

δ (shift size)	$ARL_0 = 370$		*Diff* %	$ARL_0 = 500$		*Diff* %
	Explicit	IE (CPU Time: second)		Explicit	IE (CPU Time: second)	
0.00	370.302 (0.14)	370.312 (20.936)	0.003	500.20 (0.14)	500.21 (81.217)	0.002
0.01	177.441 (0.14)	177.121 (57.473)	0.180	202.64 (0.14)	202.235 (85.21)	0.200
0.03	86.165 (0.14)	86.465 (61.451)	0.348	91.688 (0.14)	91.385 (89.219)	0.330
0.05	56.517 (0.14)	56.333 (65.413)	0.326	58.835 (0.14)	58.635 (93.151)	0.340
0.07	41.869 (0.14)	41.679 (69.344)	0.454	43.124 (0.14)	43.224 (97.082)	0.232
0.10	30.005 (0.14)	30.105 (73.307)	0.333	30.641 (0.14)	30.545 (101.09)	0.313
0.20	15.229 (0.14)	15.252 (77.239)	0.151	15.387 (0.14)	15.382 (105.05)	0.032

TABLE III. OPITMAL DESIGN PARAMITERS AND ARL_1^* FOR EWMA CHART $\alpha_0 = 1$ FOR MA(2)

α_1	$ARL_0 = 370$			α_1	$ARL_0 = 500$		
	λ	b	ARL_1^*		λ	b	ARL_1^*
1.01	0.169	0.259	96.519	1.01	0.169	0.259	103.442
1.03	0.169	0.258	39.520	1.03	0.169	0.258	40.604
1.05	0.168	0.257	25.146	1.05	0.168	0.257	25.570
1.07	0.168	0.256	18.596	1.07	0.168	0.256	18.821
1.09	0.167	0.255	14.849	1.09	0.167	0.255	14.989
1.10	0.167	0.255	13.519	1.10	0.167	0.255	13.633
1.30	0.162	0.246	5.323	1.30	0.162	0.246	5.337
1.50	0.157	0.238	3.636	1.50	0.157	0.238	3.642

TABLE IV. OPITMAL DESIGN PARAMITERS AND ARL_1^* FOR EWMA CHART $\alpha_0 = 1$ FOR MA(3)

α_1	$ARL_0 = 370$			α_1	$ARL_0 = 500$		
	λ	b	ARL_1^*		λ	b	ARL_1^*
1.01	0.113	0.228	147.452	1.01	0.112	0.228	164.347
1.03	0.113	0.229	66.708	1.03	0.113	0.229	69.908
1.05	0.113	0.230	43.018	1.05	0.113	0.230	44.305
1.07	0.113	0.231	31.721	1.07	0.113	0.231	32.404
1.09	0.114	0.232	25.121	1.09	0.114	0.232	25.540
1.10	0.114	0.232	22.756	1.10	0.114	0.232	23.097
1.30	0.115	0.236	8.074	1.30	0.115	0.236	8.110
1.50	0.115	0.236	5.130	1.50	0.115	0.236	5.143

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