

ON SUB RIEMANNIAN STRUCTURES AND RELATED MECHANICS

Sarika M. Patil^{1*}, T. VENKATESH², J.V. RAMANARAJU³

¹Department of Mathematics, Government First Grade College, Hubli,

²Department of Mathematics, Rani Channamma University, Belgaum,

³Department of Mathematics, Jain University, Bengaluru,
Karnataka State, India

Abstract— This paper deals with certain configuration spaces where the underlying geometry is sub-Riemannian because of the physical constraints arising out of the mechanical systems we are interested in. A motivated introduction to Sub-Riemannian structures is included following which we look at the broad science of mechanics where in the sub-Riemannian geometric study aids us to talk about applications like robotics and image analysis.

Index Terms— Lie group, quaternion group, sub-Riemannian, horizontal distributions

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I. INTRODUCTION

Physical dynamical systems involve a geometric space equipped with a topology and a phase space coming out of operators acting on the space. We consider the functionals arising out of the constrained motions in the Euclidean space E^3 . The dynamical systems considered here are tracking the state of the mechanical systems in question, in terms of the local coordinates. The controls applied are on the velocity vectors of the mechanical system which essentially leads to a non-integrable distribution due to a nonholonomic constrained environment. One of the main problems is to determine whether there is a continuous curve tangent to the distribution and connecting any two given points. A sub-Riemannian metric actually solves the optimization problem described here, but the challenge is to choose the correct sub-Riemannian metric among the infinite choices of such a metric. The global connectivity by special kind of curves that are tangent to the distribution can be established by the bracket generating property.

In section 1 we introduce the distributions on a sub-Riemannian manifold and mention the key examples. We also discuss the issue of optimal control in the setting of sub-Riemannian geometry. We then expound in section 2 on the interesting Lie group structures. In section 3 we come to the mechanical set-up where we apply the geometric/topological theory to some interesting problems. We conclude the exposition with several extensions of this theory.

II. SUB - RIEMANNIAN SETTING

Let (M, g) be a Riemannian manifold. Then an inner product g_p can be defined on each tangent space T_pM at all points $p \in M$, note that this implies the existence of a smooth map $p \rightarrow g_p(X(p), Y(p))$ for every pair of vector fields X, Y on M . This family of continuously varying inner products leads to a Riemannian metric on M . With this metric various geometric features on M can be described, in particular distances between two points, angles between two lines, geodesics etc. can be computed using 'g'. In many specialized

applications one however restricts tangents to certain horizontal subspaces. Here one considers a smooth vector distribution say Δ and then the metric tensor 'g' is restricted only to Δ -horizontal subspace Δ_q .

Definition 2.1(Distribution) [4]: Let $H = \text{span}\{X, Y\}$ be the distribution generated by the vector fields X and Y . Since $[Y, X] = 2T \notin H$ it follows that H is not involutive, we can write $L(s^3) = H \oplus \text{Rt}$. The distribution H will be called the horizontal distribution. Any curve on the sphere which has the velocity vector contained in the distribution H will be horizontal curve.

Such a manifold (M, Δ, g) is called a Sub-Riemannian manifold. Since the metric is restricted to the distribution Δ on M . Further the topology of a such a manifold is the one induced by the Sub-Riemannian distance given as follows.

Definition 2.2 (Sub-Riemannian Distance): Consider a Sub-Riemannian manifold (M, Δ, g) and a Lipschitzian horizontal curve $\gamma: I \subset \mathbb{R} \rightarrow M, \gamma(t) \in \Delta_{\gamma(t)}$ for almost all $t \in I$. The length of γ is given as $\text{length}(\gamma) = \int_I \sqrt{g_{\gamma(t)}(\dot{\gamma}(t))} dt$, where $g_{\gamma(t)}$ is the inner product in $\Delta_{\gamma(t)}$. The sub-Riemannian distance between two points $p, q \in M$ is length of the shortest curve joining p to q :

$$d(p, q) = \inf\{\text{length}(\gamma) : \gamma \text{ is horizontal curve, } \gamma \text{ joins } p \text{ to } q\}$$

In this exposition, we consider some applications of such a structure so as to bring out the utility of developing sub-Riemannian geometry.

III. SUB-RIEMANNIAN PROBLEM:

Consider a drift less dynamical system on Sub-Riemannian manifold (M, Δ, g) :

$$\dot{q} = \sum_{i=1}^m U_i(t) f_i(q), (u_1, \dots, u_m) \in \mathbb{R}^m.$$

The problem of finding horizontal curves γ from initial state q_0 to final state q_1 with shortest sub-Riemannian distance $d(q_0; q_1)$ and tangent to a given distribution $\Delta_g \subset T_qM$ is called sub-Riemannian problem.

Example 3.1: (Sub-Riemannian Structures)

The Lie group $SH(2)$:

The group $SH(2)$ can be represented by third order matrices:

$$M = SH(2) = \left\{ \begin{pmatrix} \cosh z & \sinh z & x \\ \sinh z & \cosh z & y \\ 0 & 0 & 1 \end{pmatrix} / x, y, z \in \mathbb{R} \right\}$$

The Lie group $SH(2)$ comprises three basis one-parameter subgroups given as:

$$w_1(t) = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}, w_2(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

mechanics, robotics and image analysis. If W_i denotes the Hamiltonian associated with the vector field X_i , then we have the following Poisson brackets. $\{W_1, W_2\} = W_3$, $\{W_1, W_3\} = W_4$, $\{W_2, W_3\} = W_3$. In optimal control problems we get control system of the form $\dot{x} = \sum u_i x_i(u)$.

Where x belongs to a configuration space which is a smooth manifold, and the functions u_i are control functions taking values in a fixed compact convex set in R^m .

The control dependent Hamiltonian that denotes the total energy is written as:

$$W(u) = -\lambda(0)/2[u_1^2] + u_1 W_1 + u_2 W_2 + \dots + u_n W_n.$$

If a trajectory $t \rightarrow g(u)$ is optimal then it is the projection of an extremal curve $t \rightarrow z(g, p)$

Satisfying $Hu(z)$ greater or equal to $Hv(z)$ for all admissible controls v .

Since the admissible controls satisfy the sub-Riemannian geometry we enter into the realm of sub Riemannian structures.

In Robotics the constraints are linear and hence we consider subriemannian spaces which have a structure of a Lie group. The Heisenberg group and $SU(2)$ are standard examples.

There are two basic problems faced by engineers namely the forward kinematic problem and the inverse kinematic problem. The very subtle problems the engineers face in such exploration are precisely inverse kinematic problems. The configuration space we considered earlier makes a presence here in the forward kinematics problem. Here we are given a configuration of the system with all possible joints and we need to arrive at the end configuration system. The configuration space QF is a Cartesian product of the spaces of individual joints of the manipulators as discussed above. Hence the forward kinematics problem is represented as $g : Q \rightarrow SE(3)$ where Q is the total configuration space and $SE(3)$ is the set of all rigid motions of the 3-dimensional Euclidean space. The mapping g is essentially a composition of rigid motions due to individual joints. We next consider the notion of a workspace W .

$W = \{g(\theta) : \theta \in Q\} \subseteq SE(3)$. The remote sensing capabilities for an automated mission are achieved by considering this subspace of the 3-dimensional Euclidean space. In the inverse kinematic problem we are given a desired end configuration and we need to build a robotic system such that the set of all controls θ_i leads to the end configuration. It is in this situation algebraic geometry is heavily used. For example in the plane a two-link manipulator has to satisfy the following equations

$$x = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \dots \dots (1)$$

$$y = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \dots \dots (2)$$

By using the 2 equations one determines answer for the forward problem essentially by solving for θ_1 and θ_2 given x and y . Thus one has a set of equations in more realistic examples and we need to determine the parameters involved.

IV. IMAGE ANALYSIS PROBLEM:

For image analysis from a neurobiological viewpoint we should be interested in measurements of images. Thus images can be parameterized by location, log-width, and orientation. While log width distinguishes various scaling of

$$w_3(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Whereas basis for Lie algebra are the tangent matrices $A_i = \frac{dw_i(t)}{dt} / t=0$ to the sub-groups of Lie group $SH(2)$.

A_i are given as :

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The Lie algebra is thus :

$$L = T_{Id}M = sh(2) = \text{span} \{A_1, A_2, A_3\}.$$

The multiplication rule for L is $[A, B] = AB - BA$.

Therefore, the Lie bracket for $sh(2)$ is given as $[A_1, A_2] = A_3$, $[A_1, A_3] = A_2$ and $[A_2, A_3] = 0$.

Example 3.2 (Sub-Riemannian structure on S^3):

We consider the parametrization of S^3 in terms of the Euler angles, which due to spherical symmetry are more suitable than the Cartesian coordinates.

Consider ϕ, ψ, θ to be the Euler's angles and let $\alpha = \frac{\phi+\psi}{2}, \beta = \frac{\phi-\psi}{2}$.

The sphere S^3 can be parametrized as

$$\begin{aligned} x_1 &= \cos \frac{\phi+\psi}{2} \cos \frac{\theta}{2} = \cos \alpha \cos \frac{\theta}{2}, \\ x_2 &= \sin \frac{\phi+\psi}{2} \cos \frac{\theta}{2} = \sin \alpha \cos \frac{\theta}{2}, \\ y_1 &= \cos \frac{\phi-\psi}{2} \sin \frac{\theta}{2} = \cos \beta \sin \frac{\theta}{2}, \\ y_2 &= \sin \frac{\phi-\psi}{2} \sin \frac{\theta}{2} = \sin \beta \sin \frac{\theta}{2}, \end{aligned}$$

with $0 \leq \theta \leq \pi, -\pi \leq \alpha \leq \pi$, In the following we shall write the restriction of the one-form to S^3 using Euler's angles. Since 1

$$\begin{aligned} dx_1 &= -\sin \alpha \cos \frac{\theta}{2} d\alpha - \frac{1}{2} \cos \alpha \sin \frac{\theta}{2} d\theta \\ dx_2 &= \cos \alpha \cos \frac{\theta}{2} d\alpha - \frac{1}{2} \sin \alpha \sin \frac{\theta}{2} d\theta \\ dy_1 &= -\sin \beta \sin \frac{\theta}{2} d\beta + \frac{1}{2} \cos \beta \cos \frac{\theta}{2} d\theta \\ dy_2 &= \cos \beta \sin \frac{\theta}{2} d\beta + \frac{1}{2} \sin \beta \cos \frac{\theta}{2} d\theta \end{aligned}$$

we obtain,

$$\begin{aligned} \omega &= x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2 \\ &= (\cos \beta \sin \alpha \sin \frac{\theta}{2} \cos \frac{\theta}{2} - \sin \beta \cos \alpha \sin \frac{\theta}{2} \cos \frac{\theta}{2}) d\alpha \end{aligned}$$

$$+ (-\cos \alpha \cos \frac{\theta}{2} \sin \beta \sin \frac{\theta}{2} + \sin \alpha \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \beta) d\beta$$

$$+ \frac{1}{2} (\cos \alpha \cos \frac{\theta}{2} \cos \beta \cos \frac{\theta}{2} + \cos \alpha \cos \beta \sin^2 \frac{\theta}{2} + \sin \alpha \sin \beta \cos^2 \frac{\theta}{2} + \sin \beta \sin \alpha \sin^2 \frac{\theta}{2}) d\theta$$

$$\begin{aligned} &= \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin(\alpha - \beta) d\alpha + \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin(\alpha - \beta) d\beta + \frac{1}{2} \cos(\beta - \alpha) d\theta \end{aligned}$$

$$= \frac{1}{2} \sin \theta \sin \psi (d\alpha + d\beta) + \frac{1}{2} \cos \psi d\theta$$

$$= \frac{1}{2} (\sin \theta \sin \psi d\phi + \cos \psi d\theta).$$

$$w/S^3 = \frac{1}{2} (\sin \theta \sin \psi d\phi + \cos \psi d\theta).$$

The constraint $\sin \theta \sin \psi d\phi + \cos \psi d\theta = 0$ is nonholonomic since

$$2dw = \cos \theta \sin \psi d\theta \wedge d\phi + \sin \theta \cos \psi d\psi \wedge d\phi - \sin \psi d\psi \wedge d\theta \neq 0.$$

The sub Riemannian structures discussed in the above examples have some very non trivial applications in

measurements orientations distinguishes the possible rotations. Thus the space of measurements $I = M \times U$, where $M = \mathbb{R}^2 \times \mathbb{R} \times S^1$ a four dimensional differentiable manifold. U is the set of all possible measurement results.

Fixing width one gets a standard circle principle bundle. Any curve in the measurement space must have its orientation and location tangents aligned. Thus with this constraint, we are led to a sub Riemannian structure on I .

Let $\Omega = dx_2 - \tan\theta dx_1 = 0 \dots (1)$ be the equation defining the constraint referred to as the cotangent equation in [1]. If we restrict our curves to this distribution respecting (1) then we get into the realm of sub Riemannian geometry.

The unit 3- sphere centered on the origin is the set of \mathbb{R}^4 defined by

$$S^3 = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \}.$$

It is often convenient to regard \mathbb{R}^4 as two complex dimensional space \mathbb{C}^2 or the space of quaternion \mathbb{H} . The unit 3- sphere is given by

$$S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1 \} \text{ or}$$

$$S^3 = \{ (q \in \mathbb{H} : |q|^2 = 1) \}.$$

The latter description represents the sphere S^3 as a set of unit quaternions and it can be considered as a group $Sp(1)$, where the group operation is just a multiplication of quaternions. The group $sp(1)$ is a 3 – dimensional Lie group, isomorphic to $SU(2)$ by the isomorphism $\mathbb{C}^2 \ni (z_1, z_2) \rightarrow q \in \mathbb{H}$. The unitary group $SU(2)$ is the group of matrices $\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$, $z_1, z_2 \in \mathbb{C}$, $|z_1|^2 + |z_2|^2 = 1$

Where the group law is given by the multiplication of matrices. Let us identify \mathbb{R}^3 with pure imaginary quaternions. The conjugation $qh\bar{q}$ of a pure imaginary quaternions h by a unit q defines rotation in \mathbb{R}^3 , and since $|qh\bar{q}| = |h|$, the map $h \rightarrow qh\bar{q}$ defines a two-to-one homomorphism $Sp(1) \rightarrow SO(3)$. The Hopf map $\pi : S^3 \rightarrow S^2$ can be defined by $S^3 \ni q \rightarrow q\bar{q} = \pi(q) \in S^2$.

The Hopf map defines a principle circle bundle also known as principal bundle. Topologically S^3 is a compact, simply connected, 3- dimensional manifold without boundary.

V. CONCLUSIONS

In this exposition we have considered simple mechanical systems where the goal is to optimize the energy functionals in such a way that the system satisfies the constraints of the possible dynamics. There several examples of physical dynamical systems like to bicycle dynamics problem with oval wheels where the description of controllability is quite subtle. Here one can view the problem in terms of kinematic controllability. One can extend the study for a class of local kinematic controllability problems and develop conditions on the vectorfields so as to achieve the desired optimization.

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