CONTINUOUS P-FRAMES AND THEIR PERTURBATION IN BANACH SPACES

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Abstract — Replacing the sequence of vectors with a net indexed by an ordered set where the set is endowed with a measure space, we obtain a generalization of discrete frames which is called continuous p-frames. The problem of combining the synthesis and analysis operators of these frames is solved in this paper. We also prove that a perturbation of a weakly measurable function \( G \) of a cp-frame \( F \) is again a cp-frame when there is a small enough gap between \( F \) and \( G \).

Index Terms — Continuous p-frames, Duality mapping, Perturbation

I. INTRODUCTION

A discrete frame is a countable family of elements in a separable Hilbert space which allows stable not necessarily unique decomposition of arbitrary elements into expansions of the frame elements. This concept was generalized by Ali, Antoine and Gazeau [1], to families indexed by an ordered set endowed with a Radon measure. These frames are known as continuous frames. For more studies about frame theory and continuous frames we refer to [1, 3, 4, 5]. We observe that various generalizations of frames have been proposed recently. Throughout this paper, \((\Omega, \mu)\) will be a measure space and \( \mu \) is a positive, \( \sigma \)-finite measure. \( X \) is a Banach space with dual \( X^* \). We choose \( 1 < p < \infty \), and \( q \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). The normed dual \( X^* \) of a Banach space \( X \) is itself a Banach space and hence has a normed dual of its own, denoted by \( X^{**} \). The mapping \( \Lambda_x : X \rightarrow X^{**} \), \( x \mapsto \Lambda_x x \) defines a unique \( \Lambda_x x \in X^{**} \) by the equation, \( \langle x, x' \rangle = \langle x', \Lambda_x x \rangle \) for each \( x' \in X^* \) and \( \| \Lambda_x x \| = \| x \| \) for each \( x \in X \). So \( \Lambda_x : X \rightarrow X^{**} \) is an isometric isomorphism of \( X \) onto a closed subspace of \( X^{**} \). If \( X \) is a reflexive Banach space then \( \Lambda_x : X \rightarrow X^{**} \) is an isometric isomorphism of \( X \) onto \( X^{**} \).

A. 2 PRELIMINARIES

Definition 2.1. A countable family \( \{ g_j \}_{j=1}^\infty \subset X^* \) is a p-frame for \( X \) if there exist constants \( A, B > 0 \) such that

\[
\left( \sum_{i=1}^{\infty} \| g_i(f) \|^p \right)^{\frac{1}{p}} \leq A \| f \|, \quad f \in X.
\]

\( \{ g_j \}_{j=1}^\infty \) is a p-Bessel sequence if at least the upper p-frame condition is satisfied.

Definition 2.2. Let \( H \) be a complex Hilbert space and \( (\Omega, \mu) \) be a measure space. The mapping \( F : \Omega \rightarrow H \) is called a continuous frame for \( H \) with respect to \((\Omega, \mu)\), if:

(i) \( F \) is weakly measurable, i.e., for each \( f \in H \), \( \omega \mapsto \langle f, F(\omega) \rangle \) is a measurable function on \( \Omega \),

(ii) There exist constants \( A, B > 0 \) such that

\[
A \| f \|^p \leq \int_{\Omega} \| \langle f, F(\omega) \rangle \|^p d\mu(\omega) \leq B \| f \|^p, \quad f \in H.
\]
\[ K^\psi(\phi) = \int \psi(\omega) \phi(\omega) d\mu(\omega) \text{ for all } \psi \in L^p(\Omega, \mu) \]
and \( \phi \in L^q(\Omega, \mu) \).

We can define the isometrical isomorphism
\[ K^\psi(\phi) = (K^\psi)^* \Lambda_q : L^q(\Omega, \mu) \rightarrow L^p(\Omega, \mu)^* \]
for which \( \Lambda_q \) is the isometrical isomorphism of \( L^q(\Omega, \mu) \) onto \( L^p(\Omega, \mu)^* \).

**Lemma 2.5.** [7]. Given a bounded operator \( U : X \rightarrow Y \), the adjoint \( U^* : Y^* \rightarrow X^* \) is surjective if and only if \( U \) has a bounded inverse on its range \( R(U) \).

### B. 3 CP-FRAMES

**Definition 3.1.** The mapping \( F : \Omega \rightarrow X^* \) is called a continuous p-frame or a cp-frame for \( X \) with respect to \((\Omega, \mu)\) if:

(i) \( F \) is weakly measurable, i.e., for each \( x \in X \), \( w \rightarrow (x,F(\omega)) = F(\omega)(x) \) is measurable on \( \Omega \).

(ii) There exist positive constants \( A \) and \( B \) such that

\[ A \| x \| \leq (\int \| x,F(\omega) \|^p d\mu(\omega))^{\frac{1}{p}} \leq B \| x \| \quad (3.1) \]

The constants \( A \) and \( B \) are called the lower and upper cp-frame bounds, respectively. \( F \) is called a tight cp-frame if \( A = B \), and a Parseval cp-frame if \( A \) and \( B \) can be chosen such that \( A = B = 1 \). \( F \) is called a cp-Bessel mapping for \( X \) with respect to \((\Omega, \mu)\), if (i) and the second inequality in (3.1) holds. In this case \( B \) is called cp-Bessel constant.

If in the definition of a cp-frame, the measure space \( \Omega = \mathbb{N} \) and \( \mu \) be the counting measure, then our cp-frame will be a p-frame and so we expect that some properties of p-frames can be satisfied in cp-frames.

Throughout this paper, we simply say \( F \) is a cp-frame for \( X \) and \( F \) is a cp-Bessel mapping for \( X \), instead of \( F \) is a cp-frame for \( X \) with respect to \((\Omega, \mu)\) and \( F \) is a cp-Bessel mapping for \( X \) with respect to \((\Omega, \mu)\), respectively.

Our study of a cp-frame is based on analysis of two operators \( U_F : X \rightarrow L^p(\Omega, \mu) \), defined by

\[ U_F x(\omega) = \langle x,F(\omega) \rangle, x \in X, \omega \in \Omega \quad (3.2) \]

and \( T_F^* : L^q(\Omega, \mu) \rightarrow X^* \) which is weakly defined by

\[ T_F^* \phi(x) = \langle x,T_F \phi \rangle = \int \phi(\omega) \langle x,F(\omega) \rangle d\mu(\omega), \phi \in L^q(\Omega, \mu), x \in X \quad (3.3) \]

It is clear that if \( F \) is a cp-Bessel mapping, then \( U_F \) is well-defined and bounded operator. \( U_F \) is called the analysis and \( T_F^* \) is called the synthesis operator of \( F \).

**Lemma 3.2.** Let \( F \) be a cp-frame for \( X \). Then the operator \( U_F : X \rightarrow L^p(\Omega, \mu) \), given by (3.2), has a closed range and \( X \) is reflexive.

**Proof.** It is easy to verify that \( U_F \) has a closed range. By the cp-frame condition, \( X \) is isomorphic to \( R(U_F) \), but \( R(U_F) \) is reflexive because it is a closed subspace of the reflexive space \( L^p(\Omega, \mu) \) and therefore \( X \) is reflexive.

**Theorem 3.3** Let \( F : \Omega \rightarrow X^* \) be a cp-Bessel mapping for \( X \) with Bessel bound \( B \). Then the operator \( T_F^* : L^q(\Omega, \mu) \rightarrow X^* \), weakly defined in (3.3), is well-defined, linear and \( \| T_F^* \| \leq B \).

**Lemma 3.4.** Let \( F : \Omega \rightarrow X^* \) be a cp-Bessel mapping for \( X \). Then:

(i) \( U_F^* = T_F(K^\psi)^{-1} \).

(ii) If \( X \) is reflexive, then \( T_F^* = K^\psi U_F^* A_X^{-1} \).

**Theorem 3.5** Let \( X \) be a reflexive Banach space and \( F : \Omega \rightarrow X^* \) be weakly measurable. If the mapping \( T_F^* : L^q(\Omega, \mu) \rightarrow X^* \) weakly defined by

\[ \langle x,T_F \phi \rangle = \int \phi(\omega) \langle x,F(\omega) \rangle d\mu(\omega), \phi \in L^q(\Omega, \mu), x \in X \]

is a bounded operator and \( \| T_F \| \leq B \), then \( F \) is a cp-Bessel mapping for \( X \).
Proof. Since $T_F$ is well-defined and bounded, for all $f \in X^*$ and $\varphi \in L^p(\Omega, \mu)$, we have
\[
\langle \varphi, T_F^* f \rangle = \langle T_F \varphi, f \rangle = \int_{\Omega} \varphi(\omega) \langle \Lambda_X^{-1} f, F(\omega) \rangle d\mu(\omega).
\]
For each $f \in X^*$, we define $\psi_f : \Omega \to C, \omega \to \langle \Lambda_X^{-1} f, F(\omega) \rangle$. Since $\psi_f$ is measurable and for each $\varphi \in L^p(\Omega, \mu)$,
\[
\left| \int_{\Omega} \varphi(\omega) \psi_f(\omega) d\mu(\omega) \right| < \infty,
\]
$\psi_f \in L^p(\Omega, \mu)$, by Theorem 2.4, we have
\[
\psi_f(\omega) = (K^p)^{-1}\langle T_F^* f, \omega \rangle, \omega \in \Omega.
\]
Hence for each $x \in X$, \[
\frac{1}{\Omega} \left| \langle x, F(\omega) \rangle \right|^p d\mu(\omega) = \left\| (K^p)^{-1} T_F^* \Lambda_X x \right\| = \left\| T_F^* \Lambda_X x \right\| \leq \left\| T_F^* \right\| \left\| x \right\|.
\]
\[
\left\| T_F^* \right\| \left\| x \right\| \leq B \left\| x \right\|.
\]
**Theorem 3.6.** Let $X$ be a reflexive Banach space and $F: \Omega \to X^*$ be a weakly measurable mapping. Then $F$ is a cp-frame for $X$ if and only if $T_F$ is a well-defined and bounded operator of $L^p(\Omega, \mu)$ onto $X^*$. In this case, the frame bounds are \[
\left\| T_F^* \right\|^{-1} \text{ and } \left\| T_F \right\|.
\]
Proof. By Theorem 3.3 and 3.5, the upper cp-frame condition satisfies if and only if $T_F$ is well-defined and bounded operator of $L^p(\Omega, \mu)$ onto $X^*$. Now suppose that $F$ is a cp-frame for $X$. Then $U_F$ has a bounded inverse on its range $R(U_F)$ and by Lemma 2.5, $U_F^*$ is surjective and therefore $T_F$ is a well-defined and bounded operator of $L^p(\Omega, \mu)$ onto $X^*$. By Lemma 3.4, for each $x \in X$,
\[
\left\| U_F x \right\| = \left\| (K^p)^{-1} T_F^* \Lambda_X x \right\| = \left\| T_F^* \Lambda_X x \right\| \leq \left\| T_F \right\| \left\| x \right\|.
\]
On the other hand since $T_F$ is bounded and surjective, $T_F^*$ is one to one, hence $T_F^*$ has a bounded inverse on $R(T_F^*)$. So by Lemma 3.4, for each $x \in X$ we have
\[
\left\| x \right\| = \left\| \Lambda_X x \right\| = \left\| (T_F^*)^{-1} T_F^* \Lambda_X x \right\| \leq \left\| (T_F^*)^{-1} \right\| \left\| U_F x \right\|.
\]
**C. 4 CP-Frame Mapping and Its Invertibility**
In this section, in order to make a cp-frame mapping, we need a mapping from the Banach space $L^p(\Omega, \mu)$ into its dual space, $L^q(\Omega, \mu)$.

**Definition 4.1.** The mapping $\phi_x$ of $X$ into the set of subsets of $X^*$, defined by
\[
\phi_x(x) = \{ x^* \in X^* : x^*(x) = \left\| x \right\|^p, \left\| x^* \right\| = \left\| x \right\| \}
\]
is called the duality mapping on $X$.

By the Hahn-Banach theorem, for each $x \in X$, $\phi_x(x)$ is nonempty and $\phi_X(0) = 0$. In general the duality mapping is set-valued, but for certain spaces it is single-valued and such spaces are called smooth.

**Definition 4.2.** Let $F: \Omega \to X^*$ be a cp-frame for $X$. The bounded mapping $S_F: X \to X^*$ defined by
\[
S_F = T_F (K^p)^{-1} \phi_L(\Omega, \mu) U_F
\]
is called a cp-frame mapping of $F$.

**Proposition 4.3.** Suppose that $F: \Omega \to X^*$ is a cp-frame for $X$ with frame bounds $A$ and $B$. Then $S_F$ has the following properties:

(i) $S_F = U_F^* \phi_L(\Omega, \mu) U_F$.

(ii) $A^2 \left\| x \right\|^2 \leq S_F(x) \leq B^2 \left\| x \right\|^2, x \in X$.

**Definition 4.4.** A mapping $[\ldots]$ from $X \times X$ into $R$ is said to be a semi-inner product on $X$ if it has these properties:

(i) $[x, x] \geq 0$ for all $x \in X$ and $[x, x] = 0$ if $x = 0$.

(ii) $[\alpha x + \beta y, z] = \alpha [x, z] + \beta [y, z]$ for all $\alpha, \beta \in R$ and for all $x, y, z \in X$.

(iii) $[x, y]^2 \leq [x, x] [y, y]$ for all $x, y \in X$.

The element $x \in X$ is called (Giles) orthogonal to the element $y \in X$ (denoted by $x \perp y$), if $[x, y] = 0$. If $M$ is a linear subspace of $X$, the notation $M^\perp$ is used to show the orthogonal complement of $M$ in Giles sense, i.e. $M^\perp = \{ x \in X; x \perp y, y \in M \}$.
Remark 4.5. Let $F: \Omega \rightarrow X^*$ be a cp-frame for $X$. Suppose that $\text{Ker}(T_F)$ and $(\text{Ker}(T_F))^\perp$ are topologically complementary in $L^0(\Omega, \mu)$, then clearly the operator $T_F|_{(\text{Ker}(T_F))^\perp}$ is invertible and $T_F^\perp = (T_F|_{(\text{Ker}(T_F))^\perp})^{-1}$ is a bounded right inverse of $T_F$.

Definition 4.6. Let $F: \Omega \rightarrow X^*$ be a cp-frame for $X$. Suppose that $\text{Ker}(T_F)$ and $(\text{Ker}(T_F))^\perp$ are topologically complementary in $L^0(\Omega, \mu)$, we define the mapping $K:X^* \rightarrow X$ by $K = \Lambda_{X}(T_F^\perp \phi_{L^0(\Omega, \mu)})T_F^\perp$.

Lemma 4.7. Let $F: \Omega \rightarrow X^*$ be a cp-frame for $X$. Suppose that $\text{Ker}(T_F)$ and $(\text{Ker}(T_F))^\perp$ are topologically complementary in $L^0(\Omega, \mu)$, then:

(i) $K(g)(\omega) \geq \frac{1}{B^2} \|g\|_{L^0(\Omega, \mu)}^2$, where $B$ denotes an upper cp-frame bound for $F$.

Moreover, when the operator $T_F^\perp T_F$ is adjoint abelian, the following assertions hold:

(ii) $S_F$ is invertible and $S_F^\perp = K$.

(iii) $S_F^\perp = U_F^\perp (K^p)^{-1} \phi_{L^0(\Omega, \mu)} T_F^\perp$.

D.5 DUALS OF CP-BESSEL MAPPINGS

In this section, $X$ is an infinite dimensional, reflexive Banach space.

Definition 5.1. [6]. A sequence $(e_i)_{i=1}^\infty$ in $X$ is called a Schauder basis of $X$, if for each $x \in X$ there is a unique sequence of scalars $(a_i)_{i=1}^\infty$, called the coordinates of $x$, such that $x = \sum_{i=1}^\infty a_i e_i$.

Let $(e_i)_{i=1}^\infty$ be a Schauder basis of a Banach space $X$.

For $j \in N$ and $x = \sum_{i=1}^\infty a_i e_i$, denote $f_j(x) = a_j$. Using Theorem 6.5 in [6], $f_j \in X^*$. The functionals $(f_j)_{i=1}^\infty$ are called the associated biorthogonal functionals (coordinate functionals) to $(e_i)_{i=1}^\infty$ and for each $x \in X$,

we have $x = \sum_{i=1}^\infty f_j(x) e_i$.

We will denote the biorthogonal functionals $(f_j)$ by $(e_i^*)$, and say that $(e_i, e_i^*)$ is a Schauder basis of $X$.

Theorem 5.2 Let $F: \Omega \rightarrow X^*$ be a cp-Bessel mapping for $X$ and $G: \Omega \rightarrow X^{**}$ be a cq-Bessel mapping for $X^*$. Then the following assertions are equivalent:

(i) For each $x \in X$, $x = \Lambda^p_{X}(T_G^\perp T_F^\perp \phi_{L^0(\Omega, \mu)}) T_F^\perp x$.

(ii) For each $g \in X^*$, $g = T_F^\perp (K^p)^{-1}T_G^\perp (\Lambda^p_{X})^\perp T_F^\perp g$.

(iii) For each $x \in X$ and $g \in X^*$, $(x, g) = \int (\langle x, F(\omega) \rangle | g, G(\omega) \rangle d\mu(\omega))$.

(iv) For each Schauder basis $(e_i, e_i^*)$ of $X$.

Error!

Definition 5.3. Let $F: \Omega \rightarrow X^*$ be a cp-Bessel mapping for $X$ and $G: \Omega \rightarrow X^{**}$ be a cq-Bessel mapping for $X^*$. We say that $(F, G)$ is a c-dual pair, if one of the assertions of Theorem 5.25, satisfies.

In this case $F$ is called a cp-dual of $G$ and by Theorem 5.2, we can say that $G$ is a cq-dual of $F$.

Definition 5.4. Let $F: \Omega \rightarrow X^*$ be a cp-frame for $X$. We say that $F$ is independent, provident that for each measurable function $\phi: \Omega \rightarrow C$ and $x \in X$,

$\int \langle x, F(\omega) \rangle | \phi(\omega) \rangle d\mu(\omega) = 0$.

implies that $\phi = 0$.

Theorem 5.5 Let $F: \Omega \rightarrow X^*$ be a cp-frame for $X$ and $\mu(E) \geq k > 0$, for each measurable set $E$, except $E = \varnothing$. Then, we have the following assertions:

(i) If $F$ is an independent cp-frame for $X$, then there exists a unique cq-frame, $G: \Omega \rightarrow X^{**}$ for $X^*$, such that $(F, G)$ is a c-dual pair.

(ii) If Ker$(T_F)$ and $(\text{Ker}(T_F))^{\perp}$ are topologically complementary in $L^0(\Omega, \mu)$, then there exists a cq-
frame $G: \Omega \rightarrow X^{**}$ for $X$, such that $(F,G)$ is a c-dual pair.

E. 6 PERTURBATION OF CP-FRAMES

Perturbation of discrete frames has been discussed in [2]. The proof of the following theorem is based on the following lemma, which was proved in [2].

**Lemma 6.1.** Let $U$ be a linear operator on a Banach space $X$ and assume that there exist $\lambda_1, \lambda_2 \in (0,1)$ such that for each $x \in X$,

$$|x - Ux| \leq \lambda_1 |x| + \lambda_2 |Ux|.$$  

Then $U$ is bounded and invertible. Moreover for each $x \in X$,

$$\frac{1 - \lambda_1}{1 + \lambda_2} |x| \leq |Ux| \leq \frac{1 + \lambda_1}{1 - \lambda_2} |x|,$$

and

$$\frac{1 - \lambda_2}{1 + \lambda_1} |x| \leq |U^{-1}x| \leq \frac{1 + \lambda_2}{1 - \lambda_1} |x|.$$  

**Theorem 6.2** Let $F$ be an independent cp-frame for $X$ and $\mu(E) \geq k > 0$, for each measurable set $E$, except $E = \emptyset$. Suppose that $G: \Omega \rightarrow X^*$ is weakly measurable and assume that there exist constants $\lambda_1, \lambda_2, \gamma \geq 0$ such that $\max(\lambda_1 + \frac{\gamma}{A}, \lambda_2) < 1$. Let for all $\phi \in L^q(\Omega, \mu)$ and $x$ in the unit sphere of $X$,

$$\left| \int_{\Omega} \phi(\omega) \langle x, F(\omega) - G(\omega) \rangle d\mu(\omega) \right| \leq \lambda \left( \int_{\Omega} \phi(\omega) \langle x, F(\omega) \rangle d\mu(\omega) \right) + \lambda \left( \int_{\Omega} \phi(\omega) \langle x, G(\omega) \rangle d\mu(\omega) \right) + \lambda \left( \int_{\Omega} \phi(\omega) \langle x, G(\omega) \rangle d\mu(\omega) \right).$$  

Then $G: \Omega \rightarrow X^*$ is a cp-frame for $X$ with bounds

$$\frac{1 - (\lambda_1 + \frac{\gamma}{A})}{1 + \lambda_1} \text{ and } \frac{1 + \lambda_2 + \frac{\gamma}{B}}{1 - \lambda_2},$$

where $A$ and $B$ are the frame bounds of $F$.

REFERENCES